

# ADVANCED SPINOR TOOLS FOR THE COMPUTATIONAL TRACKING OF NEAR-EARTH OBJECTS

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## Abstract

*In this paper Spinors of Euclidean spaces of Geometric Algebras over a real field are defined and their algebraic and rotational properties are discussed. The advantage of using Spinors in solving problems of Celestial Mechanics is illustrated by studying the tracking problem of near Earth objects. It is shown that this technique of using spinors can replace the conventional methods and also provide a richer formalism.*

**Keywords:** Euler angles, Geometric Algebra, Euclidean spaces, Rotations, Spinors.

## I. Introduction

Spinors are defined as a product of two vectors in i-plane or i-space (Hestenes, 1986). Spinors can also be viewed as elements of a minimal left ideal (Hestenes, 1966 & Lounesto, 2001). Rotations can be treated as group actions of Spinors on the vector space over which the Geometric Algebra is constructed. There are different parametrizations for Spinors in Geometric Algebra. Thus rotations also have different parametrizations depending upon the form of the Spinor considered.

Sequences of rotations play a key role in tracking near earth orbiting objects such as an aero plane or a spacecraft (Kuipers, 1999). There exists a sequence of Spinors corresponding to every sequence of rotations. The purpose of the present paper is to compare our results with the results obtained by using Quaternions and to show that Geometric Algebra works as an efficient tool to study problems in Celestial Mechanics.

## II. Geometric Algebra

Let  $E_n$  be an  $n$ -dimensional vector space over  $\mathbb{R}$ , the field of real numbers, together with a symmetric, positive definite, bilinear form  $g : E_n \times E_n \rightarrow \mathbb{R}$  denoted by  $g(x, y) = x \cdot y$ ,  $\forall x, y \in E_n$ . There exists a unique Clifford Algebra  $(C(E_n), \cdot)$  which is a universal algebra in which  $E_n$  is embedded. Henceforth, we shall identify  $E_n$  with  $\cdot(E_n)$ . We choose and fix an orthonormal basis  $B_n = \{e_1, e_2, \dots, e_n\}$  for  $E_n$ .

Let  $A_0 = \text{span} \{e_0\} = \mathbb{R}1_A = \mathbb{R}A_1 = \text{span} \{e_i\} = E_n$

And in general,  $A_k = \{ \sum_{s=1}^k a_s e_s \}$

$s \leq k$

Then  $C(E_n) = \bigoplus A_k$

Clearly  $\dim A_k \leq n C_k$  and  $\dim C(E_n) = 2^n$ .

Geometric Algebra is constructed by taking the geometric product of the vectors in the  $n$ -dimensional vector

$\square\square$  space  $E_n$ , giving multivectors as its products. The ‘Geometric Product’ of vectors denoted by  $ab$ , as

$\square\square\square\square\square$

$$ab\square = a\square \cdot \square b + a\square \wedge b\square \quad \square \quad \dots\dots\dots (1)$$

$\square\square\square\square\square$

$$ba\square = b\square \cdot a\square + b\square \wedge a\square = a\square \cdot b\square - a\square \wedge b\square \quad \dots\dots\dots (2)$$

As every element of  $C(E_n)$  is in the form  $A = A_0 \oplus A_1 \oplus \dots \oplus A_n$ , it is called a multivector. A multivector is said to be even (odd) if  $A_r = 0$  whenever  $r$  is odd (even). A detailed construction was given by Hasan (1987).

**k-blade:** Outer product of ‘k’ number of 1-vectors is called a k-blade. Note that

$e_s \square e_{i_1} e_{i_2} \dots e_{i_m}$  is a  $m$ -blade.

Define the set  $G_n = \{\pm e_s / S \subseteq N\}$ . Clearly  $G_n$  is a group with respect to the operation ‘Geometric Product’ of the elements defined by  $e_s e_T = \square(S, T) e_{S \cup T}$  with inverse of  $e_s$  and  $\{e_\square\}$  as the identity.  $G_n$  is a ‘free group’ with  $B_n$  as a finite basis.  $G_n = 2^{n+1}$ .

## 2.1 Euclidean nature of Geometric Algebra

**2.1.1 Definition: Norm of a multivector** The concept of ‘norm’ of a multivector is very important to define division in Geometric Algebra.

To every  $A \in C(E_n)$  the magnitude or modulus of  $A$  is defined as  $|A| = \sqrt{A^\dagger A}$ .

With this definition of norm,  $C(E_n)$  becomes a Euclidean algebra. The inverse of a non zero element of  $A$  of  $C(E_n)$ , is also a multivector, defined by  $A^{-1} = \frac{A^\dagger}{A^\dagger A}$ .

**2.1.2 Definition : k-space** Every k-vector  $A_k$  determines a k-space

**2.1.3 Definition: n – dimensional Euclidean space  $C(E_n)$  :** For a n-vector  $A_n$ , designate a unit n-vector ‘i’ proportional to  $A_n$ . That is  $A_n = |A_n| i$ .

‘i’ denotes the direction of the space represented by  $A_n$ .  $\square \quad \square$

**2.1.4 Definition i-space :** The set of all vectors  $x \in E_n$  which satisfy the equation  $x \wedge i = 0$ , is said to be an i-space and is denoted by  $C_n(i)$ . Such a n-vector ‘i’ is called the pseudoscalar of the plane as every other n-vector can be expressed as a scalar multiple of it.  $\square$  Note (a)  $x = x_1 \square 1 + x_2 \square 2 + \dots + x_n \square n$  is a parametric equation of the i-space.  $x_1, x_2, \dots, x_n$  are called the rectangular components of vector  $x$  with respect to the basis  $\{\square 1, \square 2, \dots, \square n\}$  (Hestenes, 1986).  $\square$  (b)  $x = x_1 \square 1 + x_2 \square 2$  is a parametric equation of the i-plane.  $x_1, x_2$  are called the rectangular components of vector  $x$  with respect to the basis  $\{\square 1, \square 2\}$ .

## 2.2 Spinors in ‘n’ dimensions

**2.2.1 Definition: Spinor** The product of two vectors in the i-space is called a Spinor.

**2.2.2 Definition: Spinor i-space :** The Spinor i-space ‘S’ is defined as

$$S_n = \{R / R = x \square y, x \square, y \square \in i\text{-space}\}$$

$S_n = C_3^+(i)$  if  $n \leq 3$ . ‘ $S_2$ ’ can be related to complex numbers and ‘ $S_3$ ’ can be related to Quaternion Algebra.

Let  $x \square = x_1 \square 1 + x_2 \square 2 + \dots + x_n \square n$  and  $y \square = y_1 \square 1 + y_2 \square 2 + \dots + y_n \square n$ .

Then the elements in the Spinor i-space ‘ $S_n$ ’ are in the form

$$\square \square \quad n \quad n \quad ( \quad )$$

$$R = x \square y = \sum x_j y_j + \sum x_j y_k - x_k y_j \square j \square k \quad j=1, k=1$$

$R$  can be written as  $\alpha + \beta i$  if  $n = 2$  or  $3$

Where  $\alpha = \sum_{j=1}^n x_j y_j$  and  $\beta = \sum_{j,k=1}^n (x_j y_k - x_k y_j)$

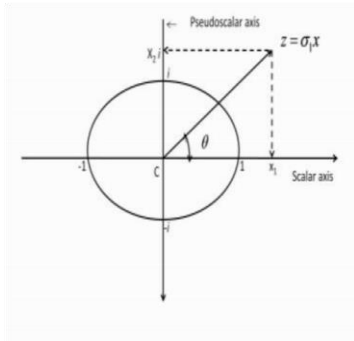


Fig 1:  $i$ - plane of Spinors

## 2.3 Algebraic Properties of Spinors

2.3.1 Theorem :  $S_n = \{ R / R = \alpha + \beta i, \alpha, \beta \in i\text{-space} \}$

- (i)  $S_n$  is an abelian group with respect to the operation  $+$  defined as the addition of the coefficients of the like terms similar to addition of polynomials.
- (ii)  $S_n$  is a vector space over  $R$ .
- (iii)  $\dim S_n = 2^{n-1}$

Spinor spaces  $S_n$  in dimensions  $n > 2$  do not satisfy commutative property. Hence they form division algebras or associative algebras.

## 2.4 Euclidean space $C_3(i)$

For a trivector  $A_3 \in C_3(i)$ , designate a unit trivector ' $i$ ' proportional to  $A_3$ . That is  $A_3 = A_3 i$ .

' $i$ ' represents the direction of the space represented by  $A_3$ .  $\square$

The set of all vectors  $x$  which satisfy the equation  $x \wedge i = 0$ , is called the Euclidean 3- dimensional vector space corresponding to ' $i$ ' and is denoted by ' $C_3(i)$ '.

$C_3(i)$  can also be called an  $i$ - space, the trivector  $i$  is called the pseudoscalar of the space as every other pseudoscalar is a scalar multiple of it.  $\square$

$x = x_1 \square 1 + x_2 \square 2 + x_3 \square 3$  is a parametric equation of the  $i$ - space where  $x_1, x_2$  and  $x_3$  are called the rectangular components of vector  $x$  with respect to the basis  $\{ \square 1, \square 2, \square 3 \}$ .  $i$ - space of vectors is a 3 – dimensional vector space with basis  $\{ \square 1, \square 2, \square 3 \}$ .

2.4.1 Definition : bivectors in  $C_3(i)$  :  $i_1 = \square 1 \square 2 = \square 3 \square 3$ ;  $i_2 = \square 2 \square 3 = \square 1 \square 1$ ;  $i_3 = \square 3 \square 1 = \square 2 \square 2$ . The set of bivectors in  $C_3(i)$  is a 3-dimensional vector space with basis  $\{ i_1, i_2, i_3 \}$ .

## 2.4.2 Spinors of Euclidean space $C_3(i)$

2.4.3 Definition: Spinor  $i$ - space The Spinor  $i$ - space ' $S_3$ ' is defined as

$\square \square \square$

$S_3 = \{ R / R = \alpha y, \alpha, y \in i\text{-space} \}$

$S_3$  can also be denoted by  $C_3^+(E)$  or  $C_3^+(i)$ . ' $R \in S_n$ ' is a multivector, has a scalar part ' $\alpha$ ' and a bivector part ' $\alpha i_1 + \alpha i_2 + \alpha i_3$ '.

TABLE 1: Comparison between Spinors of Euclidean plane and Spinors of Euclidean space

| Spinors of Euclidean plane $C_2(i)$                        | Spinors of Euclidean space $C_3(i)$                                                            |
|------------------------------------------------------------|------------------------------------------------------------------------------------------------|
| It is a Field                                              | It is an associative division algebra                                                          |
| Basis is isomorphic to the basis of $i$ -plane of vectors  | Basis is not isomorphic to the basis of $i$ -space of vectors, as it contains one more element |
| Reversion is analogous to Complex conjugation.             | Reversion is anti-isomorphic to Quaternion conjugation.                                        |
| Spinor basis is isomorphic to the basis of Complex numbers | Spinor basis is anti-isomorphic to the basis of Quaternion algebra.                            |

## 2.5 Action of Spinors on Euclidean space, Rotations

Spinors of Euclidean space also can be treated as rotation operators on  $i$ -space of vectors that is the three dimensional vector space  $E_3$ .

Unlike rotations in two dimensions, rotations in three dimensions are more complex as (i) the operation to be considered is the group action by conjugation, giving Similarity Transformations.

(ii) The axis about which the rotation takes place is also to be specified. The resulting vector changes as the axis of rotation changes. This can be shown in the following examples.

Rotate the vector  $x$  about the axis  $i_1$ , the axis perpendicular to the plane represented by the bivector  $i_1 = i_2 i_3 = i_3 i_1 = i_1 i_2$ .

Let  $x \in i$ -space of vectors and  $x = x_1 i_1 + x_2 i_2 + x_3 i_3$ .

$$i_1^\dagger x i_1 = i_3 i_2 x i_3 i_2 = i_3 i_2 (x_1 i_1 + x_2 i_2 + x_3 i_3) i_2 i_3 = x_1 i_1 - (x_2 i_2 + x_3 i_3)$$

Rotate the vector  $x$  about the axis  $i_2$ , the axis perpendicular to the plane represented by the bivector  $i_2 = i_3 i_1 = i_1 i_3$ .

$$i_2^\dagger x i_2 = i_1 i_3 x i_1 i_3 = i_1 i_3 (x_1 i_1 + x_2 i_2 + x_3 i_3) i_3 i_1 = x_2 i_2 - (x_1 i_1 + x_3 i_3).$$

## 2.6 Different parametrizations for Spinors of $C_3(i)$

### 2.6.1 Spinors of the $i$ -space in half angle form

A unit bivector is treated as a representation of the direction of an area. It can also be treated as a  $\cos \theta + i \sin \theta$  representation of an angle, which is a relation between two directions. Hence for  $x, y \in C_3(i)$ , let  $\hat{x}, \hat{y}$  be their directions which are elements of the  $i$ -space. From the definition of a Spinor of the  $i$ -space, the Spinor,  $R = \hat{x} \hat{y} = \hat{x} \cdot \hat{y} + \hat{x} \wedge \hat{y}$ .

$$= \cos \theta + i \sin \theta \quad \text{A is the half angle form of the Spinor } R.$$

1

### 2.6.2 Spinors of the $i$ -space in exponential form

$$1 \quad \hat{x} \sin \theta \quad \text{A can be written as } e^{(1/2)\theta \hat{A}}.$$

$$R = \cos \frac{A}{2} + \frac{\hat{A}}{2} \sin \frac{A}{2}$$

$$\hat{A} = \hat{x} \wedge \hat{y}$$

Here  $\hat{A} = \hat{x} \wedge \hat{y}$ , the bivector representing the plane of rotation and  $A$  gives the magnitude of the angle through which the rotation takes place.

### 2.6.3 Quaternion form

$$R = x \hat{y} = \cos \frac{A}{2} + \frac{\hat{A}}{2} \sin \frac{A}{2}$$

$$= \cos \frac{A}{2} + \frac{\hat{A}}{2} \sin \frac{A}{2}$$

$$\text{Where } \hat{A} = \hat{x} \wedge \hat{y} \text{ and } \hat{A} = \hat{x} \wedge \hat{y} = \hat{x} \wedge \hat{y}$$

It is the Quaternion form of a Spinor.

$$\hat{A} = \hat{x} \wedge \hat{y}$$

The relations between various parameters are,  $\cos \frac{A}{2} = \cos \frac{A}{2}$ ,  $\sin \frac{A}{2} = \sin \frac{A}{2}$

$$\frac{A}{2} = \frac{A}{2}$$

### 2.6.4 Euler Angle and axis form

$$\hat{A} = \hat{x} \wedge \hat{y}$$

$R = e^{(1/2) \hat{A} a}$  is the angle and axis form of the Spinor as  $\hat{a}$  is the axis of rotation and  $a$  gives the magnitude of the angle through which the given vector is rotated. This is called Euler parameterization of rotations. The parameters angle and axis are called Euler parameters.

### 2.6.5 Spinor Matrix form of a rotation

We denote a rotation by  $R$  or  $Q$  and rotation through an angle  $\theta$  by  $R_\theta$  or  $Q_\theta$ . The use of Spinors to represent a rotation gives the matrix elements directly by the formula  $e_{jk} = \frac{1}{2} \text{Tr}(R_\theta e_j e_k)$ . The advantages in using Spinors as a substitute for all the other forms for representing rotations are (i) Spinors are coordinate free.

(ii) Spinors exist in every dimension, thus make it possible to perform rotations in higher dimensional spaces also.

(iii) Spinors represent the orientation of the rotation but matrices do not.

(iv) It is easy to convert Spinors into the other forms as and when required.

### 2.7 Sequences of Spinors

Sequence or product of Spinors is also Spinor and hence a rotation. Spinors play an important role in the study of the problems related to Celestial mechanics. 2.7.1 1-2-1 symmetric sequence of rotation

We consider the 1-2-1 symmetric sequence of rotations; the Spinor that represents the required rotation is given as a sequence of three spinors about the base vectors is defined by

$$R = R_1 Q R_2 = R_1 \hat{Q} R_2 \hat{Q} R_1 \hat{Q} R_2 \hat{Q} R_1$$

$$\text{Where } R_1 = e^{(1/2) i_1 \theta_1} = \cos \frac{\theta_1}{2} + i_1 \sin \frac{\theta_1}{2}, Q = e^{(1/2) i_2 \theta_2} = \cos \frac{\theta_2}{2} + i_2 \sin \frac{\theta_2}{2},$$

$$R_2 = e^{(1/2) i_3 \theta_3} = \cos \frac{\theta_3}{2} + i_3 \sin \frac{\theta_3}{2}$$

The new set of axes after rotation are given by  $e_k = R_1 \hat{Q} R_2 \hat{Q} R_1 \hat{Q} R_2 \hat{Q} R_1$

$$\hat{Q} = \hat{Q}$$

$$= R_1 Q R_2 \hat{Q} R_1 \hat{Q} R_2 \hat{Q} R_1$$

This can be converted into the matrix form by calculating the elements of the matrix  $(e_{jk})$  given as

$$e_j^k = \delta_{jk} \cdot e_k = R_{jk}$$

$$\uparrow \quad \uparrow \quad \uparrow \quad e_1 = R_{11} =$$

$$R_{11} Q_{11} R_{11} = 1 R_{11} Q_{11} R_{11}$$

$$= R_{11} \uparrow Q_{11} \uparrow (R_{11} \uparrow 1 R_{11}) Q_{11} R_{11}$$

$$= R_{11} \uparrow (Q_{11} \uparrow 1 Q_{11}) R_{11}$$

$$= R_{11} \uparrow [1(\cos \alpha + \sin^2 \alpha)] R_{11}$$

$$= R_{11} \uparrow [1 \cos \alpha - \sin^2 \alpha] R_{11}$$

$$= (R_{11} \uparrow 1 R_{11}) \cos \alpha - (R_{11} \uparrow \sin^2 \alpha R_{11}) \sin \alpha$$

$$= 1 \cos \alpha - \sin^2 \alpha (\cos \alpha + \sin^2 \alpha) \sin \alpha$$

$$= 1 \cos \alpha + \sin^2 \alpha \sin \alpha - \sin^2 \alpha \cos \alpha$$

$$\uparrow \quad \uparrow \quad \uparrow$$

$$\text{Similarly } e_2 = R_{22} = R_{22} Q_{22} R_{22} = 2 R_{22} Q_{22} R_{22}$$

$$= 1(\sin \alpha \sin \alpha) + 2(\cos \alpha \cos \alpha - \sin \alpha \cos \alpha \sin \alpha) + 3(\sin \alpha \cos \alpha + \cos \alpha \cos \alpha \sin \alpha)$$

$$\uparrow \quad \uparrow \quad \uparrow \quad e_3 = R_{33} = R_{33} Q_{33} R_{33} = 3 R_{33} Q_{33} R_{33}$$

$$= 1 \sin \alpha \cos \alpha + 2(-\sin \alpha \cos \alpha \cos \alpha - \cos \alpha \sin \alpha) + 3(\cos \alpha \cos \alpha \cos \alpha - \sin \alpha \sin \alpha) \cos \alpha \quad (\sin \alpha \sin \alpha)$$

$$(\sin \alpha \cos \alpha) \sin \alpha \sin \alpha (\cos \alpha \cos \alpha - \sin \alpha \cos \alpha \sin \alpha) \quad (-\sin \alpha \cos \alpha \cos \alpha - \cos \alpha \sin \alpha)$$

$$-\sin \alpha \cos \alpha (\sin \alpha \cos \alpha + \cos \alpha \cos \alpha \sin \alpha) (\cos \alpha \cos \alpha \cos \alpha - \sin \alpha \sin \alpha)$$

### III. An application of sequences of spinors

#### 3.1 Tracking problem

Rotation sequences are used to track a remote object such as a spacecraft or an aero plane.

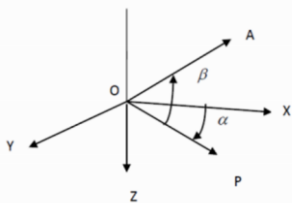


Fig 2 Tracking Problem

In the figure 3, OXYZ is the frame of reference rigidly attached to the Earth. The origin 'O' is a point on Earth from which we are observing the spacecraft.

XY plane is the Tangent plane to the Earth pointing towards North and East directions respectively. Z axis points towards the centre of the Earth (NED frame of reference). 'A' indicates the direction of the spacecraft and 'P' is the projection of A in XY plane.  $\alpha$  is the angle between the projection of the position vector of the spacecraft and the X axis.

$\beta$  is the angle between the projection of the position vector of the spacecraft and the position vector of the Spacecraft.  $\beta$  is called the 'Heading angle' and  $\alpha$  is called the 'Elevation angle' To locate the spacecraft we use  $R_{11} R_{22}$  sequence of coordinate frame transformations where  $R_{11}$  rotates the XY plane through an angle  $\alpha$  about

Z axis in such a way that the X axis coincides with the projection vector OP, and  $R_\phi$  rotates the XZ plane through an angle  $\phi$  about new y axis in such a way that

The newest 'x' axis coincides with the direction vector OA. The final direction of the X axis represents the direction of the spacecraft.

$$R = R_\phi R_\theta$$

### 3.2 Spinor Matrix form of the Tracking Transformation

The new set of axes after rotation are given by

$$\begin{matrix} \uparrow & \uparrow \\ e_k = R_\phi k = R_\phi R_\theta \sigma_k = R_\phi R_\theta \sigma_k R_\phi R_\theta \end{matrix}$$

$$\text{Where } R_\phi = e^{i(1/2)\phi\sigma_3} = \cos\frac{\phi}{2} + i\sigma_3\sin\frac{\phi}{2}, \quad R_\theta = e^{i(1/2)\theta\sigma_2} = \cos\frac{\theta}{2} + i\sigma_2\sin\frac{\theta}{2}$$

This can be converted into the matrix form by calculating the elements of the matrix  $(e_{jk})$  given as

$$e_{jk} = \sigma_j \cdot e_k = R_\phi \sigma_j$$

$$\text{Let the final set of coordinate axes be } \{e_k, k=1,2,3\}. \quad e_1 = R\sigma_1 = R_\phi R_\theta \sigma_1$$

$$= R_\phi (\sigma_1 \cos\frac{\theta}{2} + i\sigma_3 \sigma_1 \sin\frac{\theta}{2})$$

$$= R_\phi (\sigma_1 \cos\frac{\theta}{2} + i\sigma_2 \sin\frac{\theta}{2})$$

$$= \sigma_2 \cos\frac{\phi}{2} - i(\cos\frac{\phi}{2} + \sigma_3 \sigma_1 \sin\frac{\phi}{2}) \sin\frac{\theta}{2}$$

$$= \sigma_1 \cos\frac{\phi}{2} \cos\frac{\theta}{2} - \sigma_3 \sigma_1 \sin\frac{\phi}{2} \cos\frac{\theta}{2} - i\sigma_2 \sin\frac{\phi}{2} \cos\frac{\theta}{2}$$

$$= \sigma_1 \cos\frac{\phi}{2} \cos\frac{\theta}{2} - i\sigma_2 \sin\frac{\phi}{2} \cos\frac{\theta}{2} - \sigma_3 \cos\frac{\phi}{2} \sin\frac{\theta}{2}$$

Similarly

$$e_2 = R\sigma_2 = R_\phi R_\theta \sigma_2$$

$$= -\sigma_1 \sin\frac{\phi}{2} \cos\frac{\theta}{2} + i\sigma_2 \cos\frac{\phi}{2} + \sigma_3 \sin\frac{\phi}{2} \sin\frac{\theta}{2}$$

$$e_3 = R\sigma_3 = R_\phi R_\theta \sigma_3$$

$$= \sigma_3 \cos\frac{\phi}{2} + i\sigma_1 \sin\frac{\phi}{2}$$

Hence the corresponding matrix for the tracking transformation is  $\cos\frac{\phi}{2}\cos\frac{\theta}{2} -$

$$\sin\frac{\phi}{2}\cos\frac{\theta}{2} \sin\frac{\theta}{2}$$

$$\sin\frac{\phi}{2} \cos\frac{\theta}{2} \quad 0$$

$$-\cos\frac{\phi}{2}\sin\frac{\theta}{2} \sin\frac{\phi}{2} \sin\frac{\theta}{2} \cos\frac{\phi}{2}$$

### 3.3 Euler angles

Rotations transform one coordinate frame XYZ into another coordinate frame xyz preserving the angle between them. Hence it preserves the orthogonality property of the basis vectors. There is another widely used system to represent rotations is the system of Euler angles. Euler stated that every rotation can be expressed as a product of two or three rotations about fixed axes of a standard basis in such a way that no two successive rotations have the same axis of rotation. This theorem is known as 'Euler's theorem'. Thus every rotation can be divided further into two or three rotations about the fixed axes of the standard basis.

3.4 Theorem Every rotation can be expressed as a sequence of Euler angles.

Proof: We shall prove this by establishing the relation between spinor sequence of Euler angles and the angles of any arbitrary spinor sequence that represent the same rotation. As an example for an arbitrary rotation, let us

choose the symmetric 1-2-1 sequence of Euler angles obtained above. Equating the matrix representations of both we get the Euler angles in terms of  $\alpha$  and  $\beta$ .

$$R_z R_y R_x = R_z Q R_x$$

$$\begin{pmatrix} \cos\alpha & (\sin\alpha \sin\beta) & (\sin\alpha \cos\beta) \\ -\sin\alpha \cos\beta & (\sin\alpha \cos\beta + \cos\alpha \cos\beta \sin\beta) & (\cos\alpha \cos\beta \cos\beta - \sin\alpha \sin\beta) \cos\beta \cos\beta - \sin\alpha \cos\beta \sin\beta \\ \sin\alpha \cos\beta \sin\beta & \sin\alpha \cos\beta \cos\beta & \cos\alpha \cos\beta \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \sin\beta & \sin\alpha \cos\beta \\ -\sin\alpha \sin\beta & \cos\alpha \sin\beta & \cos\alpha \cos\beta \\ \sin\alpha \cos\beta & \cos\alpha \cos\beta & \cos\alpha \sin\beta \end{pmatrix}$$

$$\begin{pmatrix} \cos\alpha & (\sin\alpha \sin\beta) & (\sin\alpha \cos\beta) \\ -\sin\alpha \cos\beta & (\sin\alpha \cos\beta + \cos\alpha \cos\beta \sin\beta) & (\cos\alpha \cos\beta \cos\beta - \sin\alpha \sin\beta) \cos\beta \cos\beta - \sin\alpha \cos\beta \sin\beta \\ \sin\alpha \cos\beta \sin\beta & \sin\alpha \cos\beta \cos\beta & \cos\alpha \cos\beta \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \sin\beta & \sin\alpha \cos\beta \\ -\sin\alpha \sin\beta & \cos\alpha \sin\beta & \cos\alpha \cos\beta \\ \sin\alpha \cos\beta & \cos\alpha \cos\beta & \cos\alpha \sin\beta \end{pmatrix}$$

$$\sin\alpha \cos\beta \sin\beta = \sin\alpha \cos\beta \cos\beta$$

$$\sin\alpha \cos\beta = 0$$

$$-\cos\alpha \sin\beta \sin\beta \sin\beta \cos\beta \cos\beta = \cos\alpha \cos\beta$$

$$\tan\alpha \tan\beta =$$

$$\sin\alpha$$

$$\sin\beta$$

$$\tan\alpha = -$$

$$\tan\beta$$

These relations establish the existence of Euler angles and these relations can also be obtained using other methods also. We shall prove this by using Spinor half angle method.

### 3.5 Spinor half angle method

$$\alpha, \beta, \gamma$$

$$R_z R_y R_x = R_z Q R_x \Rightarrow \text{Let } p = 2, q = 2, r = 2, s = 2, t = 2 \text{ to avoid half angles}$$

$$(\cos q + i \sin q) (\cos p + i \sin p) = (\cos t + i \sin t) (\cos s + i \sin s) (\cos r + i \sin r)$$

$$\therefore \cos q \cos p + i \sin p \cos q + i \sin q \cos p - \sin q \sin p = \cos t \cos s + i \sin s \cos t + i \sin t \cos s - \sin t \sin s$$

$$\cos q \cos p + i \sin p \cos q + i \sin q \cos p - \sin q \sin p = \cos t \cos s + i \sin s \cos t + i \sin t \cos s - \sin t \sin s$$

$$\cos q \cos p + i \sin p \cos q + i \sin q \cos p - \sin q \sin p = \cos t \cos s + i \sin s \cos t + i \sin t \cos s - \sin t \sin s$$

$$\cos q \cos p + i \sin p \cos q + i \sin q \cos p - \sin q \sin p = \cos t \cos s + i \sin s \cos t + i \sin t \cos s - \sin t \sin s$$

$$\cos q \cos p + i \sin p \cos q + i \sin q \cos p - \sin q \sin p = \cos t \cos s + i \sin s \cos t + i \sin t \cos s - \sin t \sin s$$

$$\cos q \cos p + i \sin p \cos q + i \sin q \cos p - \sin q \sin p = \cos t \cos s + i \sin s \cos t + i \sin t \cos s - \sin t \sin s$$

$$\cos q \cos p + i \sin p \cos q + i \sin q \cos p - \sin q \sin p = \cos t \cos s + i \sin s \cos t + i \sin t \cos s - \sin t \sin s$$

$$\cos q \cos p + i \sin p \cos q + i \sin q \cos p - \sin q \sin p = \cos t \cos s + i \sin s \cos t + i \sin t \cos s - \sin t \sin s$$

$$\cos q \cos p + i \sin p \cos q + i \sin q \cos p - \sin q \sin p = \cos t \cos s + i \sin s \cos t + i \sin t \cos s - \sin t \sin s$$

$$\cos q \cos p + i \sin p \cos q + i \sin q \cos p - \sin q \sin p = \cos t \cos s + i \sin s \cos t + i \sin t \cos s - \sin t \sin s$$

$$\cos q \cos p + i \sin p \cos q + i \sin q \cos p - \sin q \sin p = \cos t \cos s + i \sin s \cos t + i \sin t \cos s - \sin t \sin s$$

Equating the coefficients of like terms we get

$$\cos q \cos p = \cos t \cos s \quad \dots \dots \dots (3) \quad \sin p \cos q$$

$$\sin p \sin q = \sin t \sin s \quad \dots \dots \dots (4) \quad \sin q \cos p = \sin s \cos t$$

$$\dots \dots \dots (5)$$

$$\sin q \sin p = \cos s \sin t \quad \dots \dots \dots (6)$$

Squaring (3) and (6) and adding

$$\cos^2 q \cos^2 p + \sin^2 q \sin^2 p = \cos^2 t \cos^2 s + \sin^2 t \sin^2 s$$

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