Statistics and Mathematical Research Journal
ISSN: 2997-6898 |
Volume 11 Issue 3, July-September, 2023
Journal Homepage: https://ethanpub.online/Journals/index.php/E32
Official Journal of Ethan Publication

EXPLORING THE APPLICATIONS OF THE LOGARITHMIC MEAN: A PRIMER ON DIFFERENCE CALCULUS

Sophie Tremblay
Department of Mathematics, Université de Montréal, Montréal, QC, Canada

Abstract:
The logarithmic mean, often referred to as the log-mean, and has proven its utility across a wide spectrum of disciplines. In this paper, we explore novel applications and uncover its potential in defining the hyperbolic function. Moreover, we introduce a novel approach, which we term the "difference calculus," for deriving two forms analogous to those produced by differential calculus. Notably, our results using this calculus can yield discrete approximations to those obtained via differential calculus. Our discussions predominantly revolve around economic data, assuming positive and discrete variables unless dealing with differentiability-driven scenarios, where continuity and differentiability are presupposed. Additionally, we primarily employ natural logarithms for simplicity. Consider two positive variables, x0 and x1, representing a base period and a comparison period, respectively. We examine their differences, Δx10 = x1 - x0, and logarithmic differences, Δlogx10 = log(x1/x0) = logx1 - logx0. We denote infinitesimal changes as dx and dlogx, with the assumption of non-zero values for interesting outcomes, even when dealing with finite changes of dependent and independent variables in certain functions.

Keywords: logarithmic mean, difference calculus, hyperbolic function, differential calculus, economic data.

1. Introduction

A logarithmic mean (hereafter log-mean) has very useful properties and has been applied over a broad range of fields [1, 3–6, 10 (see also footnotes 1 and 2 in it), 13, 20–24]. Nevertheless, there are more interesting and important areas to solve using this log-mean. For examples, it can be used to define the hyperbolic function. Moreover, whenever we employ the log-mean to decompose a function, we can easily derive two forms that are similar to those derived by the differential calculus, as we shall show later. We call this new method of derivation the "difference calculus." The results derived by our calculus may produce discrete approximations to those by the differential calculus.

In most of our discussions, most variables are assumed to be economic data, so that they are positive and discrete unless events that assume differentiability for their description are considered. (When we discuss these latter events, all variables are assumed to be continuous and differentiable.) In addition, for simplicity, they are usually not unity when we need to take their logarithms and only the natural logarithm is used.

Consider any two positive variables, x_0 and x_1, where the subscript 0 represents a base period and 1 a comparison period. Their difference is written as $\Delta x_{10} = x_1 - x_0$, and the logarithmic difference as $\Delta \log x_{10} = \log x_1 - \log x_0 = \log(x_1/x_0)$. For differentials, that is, under an infinitesimal change, they are naturally written as $\mathrm{d}x$ and $\mathrm{d}\log x$.

For finite changes, the two differences above, which include those of dependent and independent variables for some functions, are also assumed to be non-zero so as to obtain an interesting result.

Statistics and Mathematical Research Journal
ISSN: 2997-6898 |
Volume 11 Issue 3, July-September, 2023
Journal Homepage: https://ethanpub.online/Journals/index.php/E32
Official Journal of Ethan Publication

A log-mean is defined by

$$L(x) \equiv \frac{\Delta x_{10}}{\Delta \log x_{10}} = \frac{x_1 - x_0}{\log x_1 - \log x_0} = \frac{x_1 - x_0}{\log(x_1/x_0)} = \frac{x_0 - x_1}{\log(x_0/x_1)}. \qquad (1)$$

For clarity of expression, we write the log-mean (1) as $L(x)$ to explain its basic properties and applications in the two sections below. To explain further useful properties, we need to write the two arguments for the log-mean explicitly. Hence, we write the log-mean (1) as $L(x_1, x_0)$, as in Section 4. Our data are discrete, i.e., not continuous, so the integral representations of the log-mean, which may be interesting (see, for example, [8, 16, 18]), are discarded.

The remainder of this paper is organized as follows. In Section 2, we briefly explain basic properties of the log-mean. Other properties of the log-mean, which includes the relationship between this mean and the usual three means (arithmetic, geometric, and harmonic) as well as the connection between this mean and hyperbolic functions, are explained in Section 4. In Section 3, we show its applications to two areas. First, we show decompositions for some functions using two forms: an additive form and a multiplicative form. Therein, the results derived by our difference calculus that may employ the log-mean are compared with those by the conventional difference calculus (or finite-difference calculus, hereafter "conventional calculus") and the differential calculus. Second, we define a difference quotient that commonly makes the most of some log-means and show its close relationship to the differential quotient. (Remember that a form with a dependent variable given by the latter is usually called the differential equation as will be shown later.) All these aspects will make it clear that our difference calculus has many advantages over the conventional calculus, and some of the results produced by our difference calculus and quotient can be used as discrete approximations to those by the differential calculus and equation. Since a log-mean is defined for only positive variables, the scope of our difference calculus is somewhat restricted. To broaden the scope, we shall in Appendix A explain a method for handling non-positive variables; the remarks are conjectural as it has a few drawbacks. Conclusions are given in Section 5.

2. Basic Properties of a Logarithmic Mean

We briefly explain some basic properties of the log-mean (for its basic properties, see also [4–6, 8, 13, 20, 21, 23]). This value is always positive and has the limit:

$$\lim_{\Delta x_{10} \to 0} L(x) = x_1 = x_0. \qquad (2)$$

This limit induces correspondence between the difference and differential calculi to be shown later. If x_1/x_0 is close to 1, it can be approximated by the three usual means: arithmetic, geometric, and harmonic (see Subsection 4.1). This property plays an important role, when we compare the results derived by the difference and conventional calculi (see below).

Other useful properties are

$$\sum_i L(x_i) \Big/ L\left(\sum_j x_j\right) \le 1, \qquad (3)$$

$$L(x^2) = \frac{(x_1)^2 - (x_0)^2}{2(\Delta \log x_{10})} = \frac{(x_1 + x_0)(x_1 - x_0)}{2(\Delta \log x_{10})} = A(x)L(x), \qquad (4)$$

$$\sum_i L(w_i)\,\Delta \log(w_{i10}) = \sum_i \Delta w_{i10} = \sum_i (w_{i1} - w_{i0}) = 0, \qquad (5)$$

Statistics and Mathematical Research Journal
ISSN: 2997-6898 |
Volume 11 Issue 3, July-September, 2023
Journal Homepage: https://ethanpub.online/Journals/index.php/E32
Official Journal of Ethan Publication

where $A(x) = (x_1 + x_0)/2$ is the arithmetic mean; and $w_{it} = x_{it}/(\sum_j x_{jt})$, the subscript t (t = 0, 1, ...) represents a period, i (or j) the ith (or jth) commodity, and the summation is made over all commodities. Eq. (4) is often used in our difference calculus below.
When a positive x approaches zero, we have the approximation that is called the log-approximation in this paper: $\log x \approx x - 1$. This approximation yields $(x) \approx x_0 \approx x_1$, because

$$L(x) = \frac{x_0(x_1/x_0 - 1)}{\Delta \log x_{10}} = \frac{x_1(x_0/x_1 - 1)}{\Delta \log x_{01}},$$

wherein $\Delta \log x_{10} = \log(x_1/x_0) = -\log(x_0/x_1)$. See also Subsection 4.2.

3. Applications

3.1. Additive and Multiplicative Decompositions

We present the difference or ratio of a function between two periods using two forms: an additive form of which all components are additive differences and a multiplicative form of which all components are logarithmic differences. We call the former an additive decomposition (AD) and the latter a multiplicative decomposition (MD). To derive an AD and an MD for some functions, we may employ three methods: the conventional, difference, and differential calculi. One may notice that the AD and MD derived by the first two calculi may relate to those derived by the last. We explain these in this order. (There are a few functions to which the first two methods cannot be applied.) While the conventional and differential calculi are well-known, the difference calculus (i.e., our method) is less-known. Hence, we shall present many examples illustrating by our method (see also Subsection 3.2 and Appendixes A and B).
Comparing the conventional and difference calculi, we find that the former can give an AD and/or an MD for only a few functions, and the latter can apply to many functions (see also Appendix B). We would like to emphasize that the difference calculus needs to employ a log-mean to derive an AD and/or an MD.

3.1.1. Conventional Calculus

This method relates to that called the "calculus of finite differences" [7, 17]. Most differences of independent variables for a function used by the latter are unity, as in [7], which implies $\Delta X_{10} = X_1 - X_0 = (X+1) - X = 1$. Jordan [14], however, uses the following difference: $\Delta X_{t+1,} = X_{t+1} - X_t = h$, in which h is independent of t and always constant. While the conventional calculus can quickly derive an MD for any multiplicative function, it can derive an AD for only those functions composed of two independent variables as shown in Examples 1* and 2* below. (For other functions, see our extended method given in Appendix B). In contrast, this method can easily derive an AD for any additive function, but it cannot derive an MD for that as shown in Example 3* below. These derivations are also explained in the relevant text books (for example, [12, 17]).

1) Example 1*: $Y_t = X_t Z_t$.[7]

The MD of this multiplicative function is easily obtained as follows:

$\Delta \log Y_{10} = \Delta \log X_{10} + \Delta \log Z_{10}$.

Each term on the right-hand side is X's or Z's contribution to $\Delta\log Y_{10}$. The AD of this is

$Y1 - Y0 = X1Z1 - X0Z1 + X0Z1 - X0Z0 = X1Z1 - X1Z0 + X1Z0 - X0Z0$.

$\therefore \Delta Y10 = Z1\Delta X10 + X0\Delta Z10$or $\Delta Y10 = Z0\Delta X10 + X1\Delta Z10$.   (6)

Each term on the right-hand sides in Eq. (6) is X's or Z's contribution to ΔY_{10}. Whereas a convex combination of the two equations in Eq. (6) is also the AD, the arithmetic mean of those is often used as the AD, specifically

$\Delta Y_{10} = (Z)\Delta X_{10} + A(X)\Delta Z_{10}$.   (7)

Statistics and Mathematical Research Journal
ISSN: 2997-6898 |
Volume 11 Issue 3, July-September, 2023
Journal Homepage: https://ethanpub.online/Journals/index.php/E32
Official Journal of Ethan Publication

For multiple multiplicative functions, such as $Y_t = W_t X_t Z_t$, we are able to derive the AD for such functions using the more awkward procedures (see Appendix B for details).

2) Example 2*: $Y_t = X_t/Z_t$, $(X_t \neq Z_t)$.

Because the MD is easily obtained, we omit it here. The AD is

$$\Delta Y_{10} = \frac{X_1}{Z_1} - \frac{X_0}{Z_0} = \frac{X_1 Z_0 - X_0 Z_0 + X_0 Z_0 - X_0 Z_1}{Z_1 Z_0} = \frac{Z_0 \Delta X_{10} - X_0 \Delta Z_{10}}{Z_1 Z_0}$$

Comparing this decomposition with Eq. (6), we see that we have another AD. From its arithmetic mean, an AD similar to Eq. (7) is obtained:

$$\Delta Y_{10} = \frac{A(Z)\Delta X_{10} - A(X)\Delta Z_{10}}{Z_1 Z_0} \quad (8)$$

To obtain the AD of functions such as $Y_t = W_t/X_t Z_t$, we need to use the more awkward procedures presented in Appendix B.

3) Example 3*: $Y_t = X_t + Z_t$.

Since the AD for this additive function is quickly obtained, it is omitted. The MD of this is not obtained.

4) Example 7*: $Y_t = (X_t)^t$.

The conventional method can give neither an AD nor an MD for this function.

3.1.2. Difference Calculus

The difference calculus is easily able to yield ADs and MDs for many functions. Some of our methods below can be found in [5, e.g., pp. 126–132]. See also [1, 9]. Our scope is, however, wider than these.

1) Example 1: $Y_t = X_t Z_t$.

Taking the logarithm of both sides produces the MD:

$\Delta \log Y_{10} = \Delta \log X_{10} + \Delta \log Z_{10}$.

As $\Delta x_{10} = (x)\Delta \log x_{10}$, we quickly obtain its AD given by

$$\Delta Y_{10} = ((Y)/L(X))\Delta X_{10} + (L(Y)/L(Z))\Delta Z_{10}. \qquad (9)$$

If we apply the log-approximation to three log-means in Eq. (9), we have two ADs in Eq. (6). In addition, using the approximation $L(x) \approx G(x) \approx A(x)$ (see Subsection 4.1 for details), we find that Eq. (7) can be approximated by Eq. (9) (see also Appendix B).

Even if one of the independent variables is non-positive, our method may be able to derive only its AD (see Appendix A). Our approach can easily decompose more complex functions such as $Y_t = X_t Z_t W_t$ and $Y_t = (X_t)^2 (Z_t W_t)$; see examples below and Appendix B.

2) Example 2: $Y_t = X_t/Z_t$, $(X_t \neq Z_t)$.

The difference calculus quickly produces the following MD and AD:

$\Delta \log Y_{10} = \Delta \log X_{10} - \Delta \log Z_{10}$,

$$\Delta Y_{10} = ((Y)/L(X))\Delta X_{10} - (L(Y)/L(Z))\Delta Z_{10}. \qquad (10)$$

Using the approximation above, we also find that Eq. (8) can be approximated by Eq. (10).

Comparing Example 1 with Example 2, Eq. (9) with Eq. (7), and Eq. (10) with Eq. (8), we see that our calculus yields more elegant forms than those derived by the conventional calculus. For the multiplicative functions composed of three or more independent variables, see Appendix B.

3) Example 3: $Y_t = X_t + Z_t$.

The AD is $\Delta Y_{10} = \Delta X_{10} + \Delta Z_{10}$,

from which, we can derive the following MD:

$\Delta \log Y_{10} = (L(X)/L(Y))\Delta \log X_{10} + (L(Z)/L(Y))\Delta \log Z_{10}$. Note that the conventional calculus cannot derive this MD.

Statistics and Mathematical Research Journal
ISSN: 2997-6898 |
Volume 11 Issue 3, July-September, 2023
Journal Homepage: https://ethanpub.online/Journals/index.php/E32
Official Journal of Ethan Publication

4) Example 4: $Y_t = X_t + W_t Z_t$.
Letting $V_t = W_t Z_t$, we deduce its AD and MD using Example 1.
$\Delta \log V_{10} = \Delta \log W_{10} + \Delta \log Z_{10}$.
$\Delta V_{10} = ((V)/L(W))\Delta W_{10} + (L(V)/L(Z))\Delta Z_{10}$.
Using the latter equation, we produce
$\Delta Y_{10} = \Delta X_{10} + ((V)/L(W))\Delta W_{10} + (L(V)/L(Z))\Delta Z_{10}$.
$\Delta \log Y_{10} = (L(X)/L(Y))\Delta \log X_{10} + (L(V)/L(Y))(\Delta \log W_{10} + \Delta \log Z_{10})$.
The conventional calculus cannot derive either AD or MD. For the examples below, the same can also be stated.
5) Example 5: $Y_t = \exp(X_t)$.
The MD and AD are
$\Delta \log Y_{10} = L(X)\Delta \log X_{10}$,
$\Delta Y_{10} = (Y)\Delta X_{10}$.
6) Example 6: $H_t = p_t \log(1/p_t) = -p_t \log p_t > 0$ and $0 < p_t < 1$.
This function is frequently used in information theory. Here, we can employ two methods. Using $q_t = 1/p_t > 1$, we directly decompose this function (Case 1 below) and then decompose this by another method (Case 2 below). As their ADs are easily obtained from the corresponding MD, we omit the details. The two derived MDs are naturally the same.
Case 1: $H_t = p_t \log q_t$.
Taking the logarithm of both sides yields the following, from which we can derive its MD. Note that $\log q_t$ is always positive.

$$\log H_t = \log p_t + \log(\log q_t),$$

$$\therefore \Delta \log H_{10} = \Delta \log p_{10} + \frac{\Delta \log q_{10}}{L(\log q)} = \Delta \log p_{10} - \frac{\Delta \log p_{10}}{L(\log q)} = \Delta \log p_{10}\left(1 - \frac{1}{L(\log q)}\right),$$

wherein $\Delta \log q_{10} = -\Delta \log p_{10}$ and $L(\log q)$ is defined by

$$L(\log q) = \frac{q_1 - \quad q_0}{\log(\log q_1) - \log(\log q_0)} = \frac{q_{10}}{\Delta \log \log q_{10}} \text{loglog} \quad \Delta \log$$

$$\Delta \log \log q_{10} \equiv \log(\log q_1) - \log(\log q_0).$$

,
wherein
Case 2: $H_t = -p_t \log p_t$.
Squaring both sides and taking their logarithms, we have
$2\log H_t = 2 \log p_t + \log \Psi_t$,
where $\Psi_t = (\log p_t)^2$. From this equation, we obtain

$$\Delta \log H_{10} = \Delta \log p_{10} + \frac{\Psi_{10}}{2L(\Psi)} = \Delta \log p_{10} + \frac{p \ \Delta \quad p_{10}}{L(\Psi)}, \quad \Delta \ (\log \quad) \quad \log$$

$$L(\Psi) = \Delta\Psi_{10}/\Delta \log \Psi_{10},$$

$$\Delta\Psi_{10} = (\log p_1 + \log p_0)(\log p_1 - \log p_0) = 2A(\log p)\Delta \log p_{10},$$

$$L(\Psi) = L((\log p)^2) = L((\log q)^2) = A(\log q)L(\log q) = -A(\log p)L(\log q)$$

where

Recall that $A(\log q) = -A(\log p) > 0$ and $L(\log p)$ cannot be defined. Thus,

$$\Delta \log H_{10} = \Delta \log p_{10} - \frac{p_{10}}{L(\log q)} = \Delta \log p_{10}\left(1 - \frac{\Delta \log \quad 1}{L(\log q)}\right.$$

$Y_t = (X_t)^{C_t}$.

).

7) Example 7:

We show only two MDs obtained under two local (i.e., non-global) assumptions that exclude events such as $(\log X) = 0$ and $A(\log Y) = 0$. (For cases including these events, see Appendix A.) Taking the logarithm of both sides leads to

$\log Y_t = C_t \log X_t$. (11)

Assumption1: $X_t < 1$.

Here we have $\log X_t < 0$ (so $\log Y_t < 0$). Squaring both sides of Eq. (11) and taking their logarithms, we obtain $\Delta \log \Phi_{10} = \Delta \log \Gamma_{10} + \Delta \log \Psi_{10}$,

where

$\Phi_t = (\log Y_t)^2, \Delta\Phi_{10} = (\log Y_1 + \log Y_0)(\log Y_1 - \log Y_0) = 2A(\log Y)\Delta \log Y_{10}$, $\Gamma_t = (C_t)^2, \Delta\Gamma_{10} = 2A(C)\Delta C_{10} = 2A(C)L(C)\Delta \log C_{10}$, $\Psi_t = (\log X_t)^2, \Delta\Psi_{10} = 2A(\log X)\Delta \log X_{10}$.

Note that three arithmetic means are nonzero. From these and the three log-means $L(\Phi) = \Delta\Phi_{10}/\Delta \log \Phi_{10}$, $L(\Gamma) = \Delta\Gamma_{10}/\Delta \log \Gamma_{10} = L(C^2) = A(C)L(C)$, and $L(\Psi) = \Delta\Psi_{10}/\Delta \log \Psi_{10}$, we derive the MD:

$$\frac{\Delta\Phi_{10}}{L(\Phi)} = \frac{\Delta\Gamma_{10}}{L(\Gamma)} + \frac{\Delta\Psi_{10}}{L(\Psi)},$$

$$\therefore \Delta \log Y_{10} = \frac{L(\Phi)}{A(\log Y)}\left\{\Delta \log C_{10} + \frac{A(\log X)}{L(\Psi)}\Delta \log X_{10}\right\}. \qquad (12)$$

Assumption 2: $X_t > 1$.

Because $\log X_t > 0$ and $\log Y_t > 0$, we can further take the logarithm of both sides of Eq. (11) and obtain the MD:

$$\Delta \log\log Y_{10} = \Delta \log C_{10} + \Delta \log\log X_{10},$$

$$\therefore \Delta \log Y_{10} = L(\log Y)\Delta \log C_{10} + \frac{L(\log Y)}{L(\log X)}\Delta \log X_{10}, \qquad (13)$$

where $L(\log Y)$ and $L(\log X)$ are defined in a similar way to the above. In this case, we obtain $L(\Phi) = A(\log Y)L(\log Y)$ and $L(\Psi) = A(\log X)L(\log X)$.

Thus, we may also use Eq. (12), because Eq. (12) degenerates in to Eq. (13) under Assumption 2.

3.1.3. Differential Calculus

Our difference calculus is closely related to the differential calculus. The log-mean yields

$\Delta \log x_{10} = \Delta x_{10}/L(x)$. (14)

By contrast, the differential calculus produces

$\mathrm{d} \log x = \mathrm{d}x/x$. (15)

Comparing Eq. (14) with Eq. (15), we establish correspondences (a finite-change variable ↔ an infinitesimal-changevariable),

$\Delta \log x_{10} \leftrightarrow \mathrm{d} \log x$, $\Delta x_{10} \leftrightarrow \mathrm{d}x$, and $L(x) \leftrightarrow x$.

The last correspondence follows from Eq. (2).

We show the ADs and MDs for some functions using the differential calculus. These may or may not induce the correspondences above. Wherever these correspondences are found, the AD or MD derived by the difference calculus can be used as the discrete approximation to that obtained by the differential calculus.

1) Example 1:** $Y = XZ$.

$\mathrm{d}Y = Z\mathrm{d}X + X\mathrm{d}Z$. $\mathrm{dlog}\, Y = \mathrm{d} \log X + \mathrm{d} \log Z$.

Thus, $(Y)/L(X) \leftrightarrow Y/X = Z$ and $L(Y)/L(Z) \leftrightarrow Y/Z = X$.

2) Example 4:** $Y = X + WZ$. $\mathrm{d}Y = \mathrm{d}X + Z\mathrm{d}W + W\mathrm{d}Z$.

$Y\mathrm{d} \log Y = X\mathrm{d} \log X + WZ\mathrm{d} \log W + WZ\mathrm{d} \log Z$, $\therefore$ $\mathrm{d} \log Y = (X/Y)\mathrm{d} \log X + (WZ/Y)(\mathrm{d} \log W + \mathrm{d} \log Z)$.

The above-stated correspondences are evident.

3) Example 5:** $Y = \exp(X)$ $\mathrm{d}Y = \exp(X)\mathrm{d}X = Y\mathrm{d}X$. $\mathrm{d} \log Y = X\mathrm{d} \log X$.

Again, the foregoing correspondences are evident.

4) Example 7:** $Y = X^C$.

We show only the MD,

$\mathrm{d} \log Y = \log X\, \mathrm{d}C + C\mathrm{d} \log X = (C \log X)\, \mathrm{d} \log C + C\mathrm{d} \log X$, which holds for any positive X. Hence, the MD does not necessarily have the correspondences stated above, which does only if $X > 1$ (compare the above with Eq. (12) and Eq. (13)).

3.2. Difference Calculus for a Complex Function

There is a complex function having an additive-multiplicative term such as $(x+w)/z$ or $(z - w)^2 = (z - w)(z - w)$. To apply our difference calculus to this function, we may need to transform the variables. If such transformation of variables must be considered, the difference calculus may have multivalent ADs and MDs. On the contrary, the differential calculus always produces a univalent AD and MD. Our multivalent ADs and MDs, however, produce the same values as these univalent AD and MD, if all the changes in variables approach zero. To simplify the discussion in what follows, we show some examples using only ADs without the subscript t.

1) Example 8: $Y = X(Z - W) = XF = G - H$.

Here, $F = Z - W$, $G = XZ$, and $H = XW$. The differential calculus leads to the AD given by $\mathrm{d}Y = (Z - W)\mathrm{d}X + X\mathrm{d}Z - X\mathrm{d}W$.

Recalling our assumptions (i.e., Y, X, Z, and W are positive; so $F > 0$), the difference calculus provides two ADs. 1) With $F = Z - W$, the AD is obtained from

$\Delta \log Y = \Delta \log X + \Delta \log F$, $\Delta F = \Delta Z - \Delta W$.

$$\therefore \Delta Y = \frac{(\ )}{L(X)}\Delta X + \frac{(\ )}{L(F)}\Delta F = \frac{(\ )}{L(X)}\Delta X + \frac{(\ )}{L(F)}\Delta Z - \frac{(\ )}{L(F)}\Delta W. \quad L\,Y \quad L\,Y \quad L\,Y \quad L\,Y \quad L\,Y$$

(16)

2) With $G = XZ$ and $H = XW$, the AD follows from equations:

$= \quad - \quad , \quad = \quad + \quad , \quad = \quad + \quad , \quad \Delta Y \ \Delta G \quad \Delta H\ \Delta \log G \quad \Delta \log X\ \Delta \log Z$

$$\therefore \Delta Y = \left(\frac{L(G)}{L(X)} - \frac{L(H)}{L(X)}\right)\Delta X + \frac{L(G)}{L(Z)}\Delta Z - \frac{L(H)}{L(W)}\Delta W. \qquad (17)$$

$\Delta \log H \qquad \Delta \log X \Delta \log W$

These separate contributions to ΔY from X, Z, and W in Eq. (16) and Eq. (17) are different. The question now arises: which set of contributions should be selected as the best. We want to leave the matter open. (In the example below, a similar problem arises.)

If, however, the changes in each variable approach zero, each contribution approaches the differential results:

Statistics and Mathematical Research Journal
ISSN: 2997-6898 |
Volume 11 Issue 3, July-September, 2023
Journal Homepage: https://ethanpub.online/Journals/index.php/E32
Official Journal of Ethan Publication

$$\frac{L(Y)}{L(X)} \leftrightarrow Z - W, \frac{L(G)}{L(X)} - \frac{L(H)}{L(X)} \leftrightarrow Z - W, \frac{L(Y)}{L(F)} \leftrightarrow X, \frac{L(G)}{L(Z)} \leftrightarrow X, \frac{L(H)}{L(W)} \leftrightarrow X.$$

This property also holds in the example below.

2) Example 9 (saving ratio): $b = (Y - C)/Y = S/Y = 1 - a$.

Here, $S = Y - C$ and $a = C/Y$; and the quantities Y, C, S, a, and b are, respectively, income, consumption, saving, propensity to consume, and saving ratio. For dissaving (i.e., $S < 0$ and $b < 0$), see the discussion in Appendix A. The differential calculus yields:

$$\mathrm{d}b = \frac{C\mathrm{d}Y}{Y^2} - \frac{\mathrm{d}C}{Y}.$$

The difference calculus yields two ADs.

1) For $b = S/Y$,

$$\Delta \log b = \Delta \log S - \Delta \log Y, \Delta S = \Delta Y - \Delta C,$$

$$\therefore \Delta b = \left(\frac{L(b)}{L(S)} - \frac{L(b)}{L(Y)}\right)\Delta Y - \frac{L(b)}{L(S)}\Delta C.$$

$$\Delta b = -\Delta a, \Delta \log a = \Delta \log C - \Delta \log Y,$$

2) For $b = 1 - a$,

$$\therefore \Delta b = \frac{L(a)}{L(Y)}\Delta Y - \frac{L(a)}{L(C)}\Delta C.$$

3.3. Difference Quotient versus Differential Quotient

As $\mathrm{d}Y/\mathrm{d}X$ is sometimes called the differential quotient, we call $\Delta Y/\Delta X$ a "difference quotient." We recall the natural fundamental definition of the differential quotient:

$$\lim_{\Delta X \to 0} \frac{\Delta Y}{\Delta X} \equiv \frac{\mathrm{d}Y}{\mathrm{d}X}.$$

The result derived by the differential quotient, which will be explained below, is the differential equation. Thus, that derived by the difference quotient is regarded as the discrete approximation to this equation from the definition above. Considering the correspondences shown in Subsubsection 3.1.3, we are able to find a situation wherein the difference quotient corresponds to the differential one. Inspired by Subsection 3.2, we come to the understanding that not all functions can yield the difference quotients that directly correspond to their differential versions. Moreover, our method of deriving difference quotients does not apply to some functions such as trigonometric functions, because the arguments for the log-mean must be positive. We now discuss some examples. In the following, we omit the subscripts t on all functions and their difference forms to stress these correspondences.

1) $Y = (X)^n$, where n is a non-zero constant.

$\Delta \log Y = \Delta Y/L(Y) = n\Delta \log X = n(\Delta X/L(X))$.

Thus, the difference quotient is

$$\frac{\Delta Y}{\Delta X} = n\frac{L(Y)}{L(X)}.$$

In contrast, we have the following differential quotient, the form of which is typical of a differential equation,

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = nX^{n-1} = n\frac{X^n}{X} = n\frac{Y}{X}.$$

Statistics and Mathematical Research Journal
ISSN: 2997-6898 |
Volume 11 Issue 3, July-September, 2023
Journal Homepage: https://ethanpub.online/Journals/index.php/E32
Official Journal of Ethan Publication

Provided that ΔX approaches 0, as in the definition above, the difference and differential quotients are close. Besides, replacing $L(z)$ with z, we can obtain the right hand of the latter from that of the former. Thus, the result produced by the former is also regarded as the discrete approximation to the differential equation. (The same property holds in the examples below.)

2) Y = exp(X).

Example 5 and Example 5** in Subsection 3.1 produce

$$\frac{\Delta Y}{\Delta X} = L(Y) \text{ and } \frac{dY}{dX} = \exp(X) = Y.$$

3) $Y = a^X$, $a > 0$ is constant.

$$\Delta \log Y = \Delta X(\log a), \frac{\Delta Y}{\Delta X} = L(Y)\log a, \text{ and } \frac{dY}{dX} = Y \log a.$$

Compare our difference quotient with Boole's [7, p. 10].[11]

4) **Y = FG**, where $F = (X)^2 + a > 0$, $G = X + b > 0$; and a and b are constants.

$\Delta \log Y = \Delta \log F + \Delta \log G$,

$$\Delta Y = (L(Y)/L(F))\Delta F + (L(Y)/L(G))\Delta G. \quad (18)$$

Setting $C = (X)^2$, we have $\Delta \log C = 2\Delta \log X$. Hence,

$$\Delta F = \Delta C = 2((C)/L(X))\Delta X. \quad (19)$$

Substituting Eq. (19) and $\Delta G = \Delta X$ into Eq. (18) yields

$$\Delta Y = 2(L(Y)/L(F))(L(C)/L(X))\Delta X + (L(Y)/L(G))\Delta X,$$

$$\therefore \frac{\Delta Y}{\Delta X} = \frac{2L(Y)L(X^2)}{L(F)L(X)} + \frac{L(Y)}{L(G)}.$$

In contrast, the differential quotient is

$$dY = GdF + FdG = 2GXdX + FdX,$$

$$\therefore \frac{dY}{dX} = 2GX + F = \frac{2YX^2}{FX} + \frac{Y}{G}.$$

4. Further Properties of the Logarithmic Mean

4.1. Comparison of the Logarithmic Mean with the Usual Three Means

In this section, we explicitly write the two arguments for the logarithmic, arithmetic, geometric, and harmonic means, specifically, $L(x_1, x_0)$, $A(x_1, x_0)$, $G(x_1, x_0)$, and $H(x_1, x_0)$ for two positive variables, x_1 and x_0. These are also rewritten as $x_0L(x_1/x_0, 1)$, $x_0A(x_1/x_0, 1)$, $x_0G(x_1/x_0, 1)$, and $x_0H(x_1/x_0, 1)$. Hence, we have $A(x_1, x_0)/L(x_1, x_0) = A(x_1/x_0, 1)/L(x_1/x_0, 1)$, $G(x_1, x_0)/L(x_1, x_0) = G(x_1/x_0, 1)/L(x_1/x_0, 1)$, etc. For a positive constant c, all the means have the following properties:

$$L(cx_1, cx_0) = \frac{cx_1 - cx_0}{\log(x_1/x_0)} = \frac{c(x_1 - x_0)}{\log(x_1/x_0)} = cL(x_1, x_0), A(cx_1, cx_0) = cA(x_1, x_0),$$

$$G(cx_1, cx_0) = cG(x_1, x_0), \text{ and } H(cx_1, cx_0) = cH(x_1, x_0).$$

When the absolute value w is very small[13], we obtain

$$L(1 + w, 1) = \frac{w}{\log(1 + w)} \approx 1/\left(1 - \frac{1}{2}w + \frac{1}{3}w^2\right),$$

wherein we used $\log(1 + w) \approx w - (1/2)w^2 + (1/3)w^3$, and

$$G(1 + w, 1) = (1 + w)^{0.5} \approx 1 + \frac{1}{2}w - \frac{1}{8}w^2.$$

Statistics and Mathematical Research Journal
ISSN: 2997-6898 |
Volume 11 Issue 3, July-September, 2023
Journal Homepage: https://ethanpub.online/Journals/index.php/E32
Official Journal of Ethan Publication

By contrast, the arithmetic mean is

$$A(1+w,1)=1+\frac{w}{2}.$$

Hence, we find

$$\frac{G(1+w,1)}{L(1+w,1)}\approx 1-\frac{1}{24}w^2,$$

$$\frac{A(1+w,1)}{L(1+w,1)}\approx 1+\frac{2}{24}w^2.$$

These approximations lead to

$$\frac{G(1+w,1)}{A(1+w,1)}\approx 1-\frac{3}{24}w^2,$$

and

$$2\frac{G(1+w,1)}{L(1+w,1)}+\frac{A(1+w,1)}{L(1+w,1)}\approx 3,$$

$$\therefore L(1+w,1)\approx \left(2G(1+w,1)+A(1+w,1)\right)/3$$

.

The last approximation is the same as that given in [8, 15].[1] Because the ratio of the harmonic mean to the log-mean is given by

$$\frac{H(1+w,1)}{L(1+w,1)}=\left(\frac{G(1+w,1)}{L(1+w,1)}\right)^2\left(\frac{L(1+w,1)}{A(1+w,1)}\right),$$

we have the following approximation using the two approximations given above:

$$\frac{H(1+w,1)}{L(1+w,1)}\approx 1-\frac{4}{24}w^2.$$

Therefore, the three approximations lead to

$(1 + w, 1) \le G(1 + w, 1) \le L(1 + w, 1) \le A(1 + w, 1)$

If the absolute value of $\Delta x_{10}/x_0$ is very small, we can utilize these approximations in the foregoing ratios such as $A(x_1, x_0)/L(x_1, x_0)$ because $x_1/x_0 = 1+\Delta x_{10}/x_0$. Thus, the log-mean may be approximated by the usual three means. For the actual errors produced by the approximations, see [21, Table 1]. These approximations are handy, when we compare the results derived by the difference and conventional calculi (see Appendix B).

The usual three means have helpful properties given by

$A(A(a, b), A(c, d)) = A(A(a, c), A(b, d)) = A(A(a, d), A(b, c)) = A(a, b, c, d)$,
$G(G(a, b),G(c, d)) = G(G(a, c), G(b, d)) = G(G(a, d), G(b, c)) = G(a, b, c, d)$,
$H(H(a, b),H(c, d)) = H(H(a, c), H(b, d)) = H(H(a, d), H(b, c)) = H(a, b, c, d)$,

where a, b, c, and d are positive variables, $A(A(a, b), A(c, d))$ is the arithmetic mean for any paring, $A(a, b, c, d)$ is that for four variables, and so on. Conversely, the log-mean does not always retain this property. We present a simple example. Given that $a = 1$, $b = 2$, $c = 3$, and $d = 4$, we have

$L(L(a, b), L(c, d)) = 2.31225\ldots$, $L(L(a, c), L(b, d)) = 2.31221\ldots$, $L(L(a, d), L(b, c)) = 2.31188\ldots$.
$\therefore L(L(a, b), L(c, d)) \neq L(L(a, c), L(b, d)) \neq L(L(a, d), L(b, c)) \neq L(L(a, b), L(c, d))$.

These pedantic derivations are very difficult, because we must calculate, for example, the following $x - y$ or x/y: $L(a, b) = 1/\log 2$, $L(c, d) = 1/\log(4/3)$, and $L(L(a, b),L(c, d)) = x$; $L(a, c) = 2/\log 3$, $L(b, d) = 2/\log 2$, and $L(L(a, c), L(b, d)) = y$.

[1] We also find that the Heron mean for $p = 1$ defined by $(G(1 + w, 1) + 2A(1 + w, 1))/3$ is greater than or equal to the log-mean (see [13]).

Statistics and Mathematical Research Journal
ISSN: 2997-6898 |
Volume 11 Issue 3, July-September, 2023
Journal Homepage: https://ethanpub.online/Journals/index.php/E32
Official Journal of Ethan Publication

Note that these logarithms are transcendental (i.e., irrational) numbers. From these inequalities and the results shown in Section 3 and Subsection 4.3 below, we infer that the log-means for three or more variables (e.g., those defined by [16, 18]) have no practical meaning.

4.2. Log-Approximation Error

Whenever a positive value of X is close to 1, we may use the log-approximation given by $\log X \approx X - 1$. We shall explain this error ratio, which is given by

$$\frac{X-1}{\log X} - 1 = L(X,1) - 1 \approx \frac{X+1}{2} - 1 = \frac{X-1}{2},$$

wherein we have used $L(X, 1) \approx A(X, 1)$. If $X = 1$, the ratio is null. If X lies between 0.95 and 1.05, this ratio lies within about ±2.5 percent.

4.3. Novel Forms and Induced Formulae

The log-mean can be stated in novel ways that yield some useful formulae. We begin by presenting two of them:

$$\frac{1}{L(X_1,X_0)} = \frac{\log(X_1/X_0)}{X_1 - X_0}, \therefore \exp\left(\frac{1}{L(X_1,X_0)}\right) = \left(\frac{X_1}{X_0}\right)^{1/(X_1-X_0)},$$

$$L(1+w,1) = \frac{w}{\log(1+w)}, \therefore \exp\left(\frac{1}{L(1+w,1)}\right) = (1+w)^{1/w},$$

for which $X_1 > 0$, $X_0 > 0$, and $w > -1$. Thus, the three formulae are obtained:

$$\lim_{X_1 \to X_0} \left(\frac{X_1}{X_0}\right)^{1/(X_1 - X_0)} = e^{1/X_0},$$

$$\lim_{w\to -0}(1+w)^{1/w} = e, \lim_{w \to +0}(1+w)^{1/w} = e$$

.

The last two tell us the following formulae, one of which is well-known:

if $-1 < w = 1/n \le 0$, then $e = \lim_{n\to-\infty}\left(1+(1/n)\right)^n$,

if $0 \le w = 1/n$, then $e = \lim_{n\to\infty}\left(1+(1/n)\right)^n$.

The log-mean also allows us to gain new insights into the nature of hyperbolic functions. We present next a

few of them, from which we can obtain other useful formulae. The log-means connected with the hyperbolic sine:

$$L(e^x, e^{-x}) = \frac{e^x - e^{-x}}{2x}, \therefore \sinh x = \frac{e^x - e^{-x}}{2} = xL(e^x, e^{-x})$$

In addition, the hyperbolic cosine is

$$\cosh x = \frac{e^x + e^{-x}}{2} = A(e^x, e^{-x}) = \frac{L(e^{2x}, e^{-2x})}{L(e^x, e^{-x})},$$

where we have used (4). Hence

$(\sinh x)(\cosh x) = x(e^{2x}, e^{-2x})$.

These formulae hold for any real x. If we use a new variable z defined by $x = \log z$, simpler formulae are gained: $(\sinh x)/x = L(z, 1/z) = L(z^2,1)/z$, $\cosh x = A(z, 1/z) = A(z^2, 1)/z$,

$\therefore (z, 1/x) + A(z, 1/z) = (\sinh x)/x + \cosh x$.

From these results, we can reformulate many of the well-known formulae for hyperbolic functions in terms of the log-means. As an example, consider
$(\sinh x)(\cosh y) = (\sinh(x + y) + \sinh(x - y))/2$,
from which we have
$xL(e^x,e^{-x})A(e^y,e^{-y}) = ((x + y)L(e^{x+y},e^{-x-y}) + (x - y)L(e^{x-y},e^{-x+y}))/2$.

4.4. Logarithmic Mean for $x_t \pm k$

It is well-known that the log-mean for cx_1 and cx_0, with any positive constant c, is given by $cL(x_1, x_0)$. What is
it for $x_1 \pm c$ and $x_0 \pm c$? When we know only the range of x_t that is generally given by $[x_t - d, x_t + d]$ for a small positive d, we may want to examine the range of the log-mean $L(x_1, x_0)$. If $k \neq 0$ is any constant (positive or negative) and its absolute value is denoted $|k| \ll \text{Min}\{x_0,x_1\}$, we establish a nice relation between $L(x_1, x_0)$ and $L(x_1 \pm k, x_0 \pm k)$. Here, $x_t \pm k$ is always positive and a very small $|k|$ is desirable.
The latter log-mean is

$$\begin{aligned}\mathrm{L}(x_1+k, x_0+k) &= \frac{x_1 - x_0}{\log\big(x_1(1+k/x_1)\big) - \log\big(x_0(1+k/x_0)\big)} \\ &\approx \frac{x_1 - x_0}{\log(x_1/x_0) + k(1/x_1 - 1/x_0)},\end{aligned}$$

wherein we used the log-approximation, $\log(1 + k/x_t) \approx k/x_t$. In contrast, we have another log-mean that is given by

$$\frac{L(x_1, x_0)}{x_1 x_0} = L(1/x_1\,, 1/x_0) = \frac{1/x_1 - 1/x_0}{-\log(x_1/x_0)}.$$

Thus, we derive

$$L(x_1+k, x_0+k) \approx \frac{L(x_1, x_0)}{1 - (k/(x_1 x_0))L(x_1, x_0)}. \quad (20)$$

If $d \to |k|$, the range of the log-mean $L(x_1, x_0)$ is

$(x_1 - |k|, x_0 - |k|) < L(x_1, x_0) < L(x_1 + |k|, x_0 + |k|)$;
Accordingly, this lower and upper bounds can be computed using $L(x_1, x_0)$. If we can moreover assume $(x_1, x_0) \approx$
(x_1, x_0), we obtain from Eq. (20)

$$\frac{L(x_1, x_0)}{L(x_1+k, x_0+k)} \approx 1 - \frac{k}{G(x_1, x_0)}.$$

5. Conclusion

We have shown useful properties of the logarithmic mean (log-mean) and applied it to decompositions for some functions. We have also established that this mean can be approximated by the three usual means (arithmetic, geometric, and harmonic means) and given by the hyperbolic functions.
Our discussions are mainly focused on two areas. One is decomposing the difference or ratio of a function between two periods into an additive form of which all components are additive differences and a multiplicative form of which all components are logarithmic differences. We call the former an additive decomposition (AD) and the latter a multiplicative decomposition (MD). To derive an AD and/or an MD for some functions, our method called the difference calculus needs to employ a log-mean. Using this method,

Statistics and Mathematical Research Journal
ISSN: 2997-6898 |
Volume 11 Issue 3, July-September, 2023
Journal Homepage: https://ethanpub.online/Journals/index.php/E32
Official Journal of Ethan Publication

we have derived the ADs and MDs for many functions, and compared some results with those derived by the conventional (finite-difference) and differential calculi.
The other area involves illustrating the close correspondences between the difference quotient that is commonly given by a ratio of some log-means and the differential quotient that is generally called a derivative. We have explained these correspondences using various functions and presented a certain function without the correspondence.
In these discussions, we have demonstrated the following three points: 1) our difference calculus has many advantages over the conventional calculus to derive ADs and MDs for some functions; 2) some of the results obtained by our calculus can/cannot be used as discrete approximations to those by the differential calculus; and 3) some expressions produced by the difference quotients can/cannot be used as discrete approximations to corresponding differential equations. In particular, the latter two points are important wherever we must derive a proper discrete approximation to a differential calculus or a differential equation in the information age (see also online [11: "Finite-difference calculus, Computational mathematics, and Numerical analysis"]).

Appendix A: Difference Calculus for Non-Positive Variables

As logarithms are only defined for positive variables, our difference calculus cannot derive an MD for a function depending on a non-positive variable. Nevertheless, an AD may be derivable for it. Using a simple function such as Example 1 in Subsubsection 3.1.2, we shall derive two ADs: one including a non-zero variable and another including a non-positive variable (see also Examples 6 and 7 in that subsubsection).We explain the methods to handle these problems as conjectures, because we need one more assumption for the former AD and we are unable to derive a definite result for the latter AD. In this appendix, we assume that only one of Z_t (t = 0, 1) may be non-positive, so $Z_1 + Z_0$ and one of Y_t (t = 0, 1) may be zero. (Recall the two differences of X_t and Z_t are non-zero under the initial assumptions.)

Method 1: $Y_t = X_t Z_t$ and $Z_t \neq 0$.

Squaring both sides of this function leads to
$\Phi_t = \Psi_t \Omega_t$, wherein $\Phi_t = (Y_t)^2, \Psi_t = (X_t)^2$, and $\Omega_t = (Z_t)^2$. Here, we must assume $\Phi_1 \neq \Phi_0$ (i.e. , $Y_1 \neq -Y_0$), because we
use its two differences. (Remember that $Y_1 \neq Y_0$ under the initial assumption.) Thus we have

$\Delta \log \Phi_{10} = \Delta \log \Psi_{10} + \Delta \log \Omega_{10}$,

$$\therefore \frac{\Delta\Phi_{10}}{L(\Phi)} = \frac{\Delta\Psi_{10}}{L(\Psi)} + \frac{\Delta\Omega_{10}}{L(\Omega)}. \qquad \text{(A1)}$$

The six terms in the above are given by

$\Delta\Phi_{10} = (Y_1 + Y_0)(Y_1 - Y_0) = 2A(Y)\Delta Y_{10}, \Delta\Psi_{10} = 2A(X)\Delta X_{10}, \Delta\Omega_{10} = 2A(Z)\Delta Z_{10}$,
$(\Phi) = \Delta\Phi_{10}/\Delta \log \Phi_{10}$, $L(\Psi) = \Delta\Psi_{10}/\Delta \log \Psi_{10}$, $L(\Omega) = \Delta\Omega_{10}/\Delta \log \Omega_{10}$.

Since $Z_1 + Z_0 = 0$ may hold, $\Delta\Omega_{10}$ and $\Delta\log \Omega_{10}$ may be null. If so, $(\Omega) = \Omega_1 = \Omega_0$. Whereas Eq. (A1) is strictly not derived, we may externally regard Eq. (A1) as an identity because $\Delta\Omega_{10} = 0$ in there. Thus our AD is

$$\Delta Y_{10} = \frac{L(\Phi)}{A(Y)}\left\{\frac{A(X)}{L(\Psi)}\Delta X_{10} + \frac{A(Z)}{L(\Omega)}\Delta Z_{10}\right\}, \qquad \text{(A2)}$$

wherein $A(Z)$ may be null and $A(Y) \neq 0$ is crucial.
If $Z_t > 0$ (i. e. , $A(Z) \neq 0$ and $Y_t > 0$), we obtain the following from Eq. (4):
$L(\Phi) = A(Y)L(Y)$, $L(\Psi) = A(X)L(X)$, and $L(\Omega) = A(Z)L(Z)$. Therefore, the A D of Eq. (A2) degenerates to Eq. (9) under this assumption.

Statistics and Mathematical Research Journal
ISSN: 2997-6898 |
Volume 11 Issue 3, July-September, 2023
Journal Homepage: https://ethanpub.online/Journals/index.php/E32
Official Journal of Ethan Publication

Method 2: $Y_t = X_t Z_t$ and $Z(t = 0 \text{ or } 1)$ is any variable including a non-positive one.
One of Z_t $(t = 0, 1)$ may be null. First, we set a positive constant c that must satisfy the following:
$c \ll \mathrm{Min}\{|Z_0|,|Z_1|\}$ for $Z_t \neq 0$ $(t = 0, 1)$,
or $c \ll |Z_s|$ for $Z_s \neq 0$ and $Z_{t\neq s} = 0$ $(s = 0, 1)$.
A very small c is desirable, so that the conditions, to be mentioned below, are satisfied (see also Table 2 below). For example, if $\{Z_0 = -0.5,_1 = 0.2\}$,then $c = 10^{-10}$; and if $\{Z_0 = 2, Z_1 = 0\}$,then $c = 10^{-8}$. Using this c, we define two new variables:
$U_t = (Z_t + c)/2 \neq 0$, $V_t = (Z_t - c)/2 \neq 0$ and $Z_t = U_t + V_t$.
Thus
$Yt = XtZt = XtUt + XtVt = Mt + Nt, \Delta Y10 = \Delta M10 + \Delta N10,$
in which $M_t = X_t U_t = (Y_t + cX_t)/2 \neq 0$ and $N_t = X_t V_t = (Y_t - cX_t)/2 \neq 0$. From these, we have $M_1 + M_0 = (Y_1 + Y_0 + (X_1 + X_0))/2$ and $M_1 - M_0 = (Y_1 - Y_0 + c(X_1 - X_0))/2$.
We can obtain two values, $c1$ and $c2$, from the two equations above,
$c1 = -(Y_1 + Y_0)/(X_1 + X_0)$, $c2 = -(Y_1 - Y_0)/(X_1 - X_0)$.
Our c is sufficiently small as to satisfy the following conditions:
$c \ll |c1|$ and $c \ll |c2|$.
Hence, we have $|M_1| \neq |M_0|$ (i.e., $M_1 \neq \pm M_0$). Similarly, we have $|N_1| \neq |N_0|$.
Next, we apply the foregoing procedures used in Method 1 to these functions M_t and N_t. From M_t, we obtain the following: $\Gamma_t = \Psi_t \Theta_t$, wherein $\Gamma_t = (M_t)^2$ and $\Theta_t = (U_t)^2$ (Ψ_t was defined by the above),

$$\therefore \frac{\Delta\Gamma_{10}}{L(\Gamma)} = \frac{\Delta\Psi_{10}}{L(\Psi)} + \frac{\Delta\Theta_{10}}{L(\Theta)}.$$

Two log-means are defined in like manner to the above, and the other two terms are given by
$\Delta\Gamma_{10} = 2(M)\Delta M_{10}$ and $\Delta\Theta_{10} = 2A(U)\Delta U_{10}$.
Hence, we derive

$$\Delta M_{10} = \frac{L(\Gamma)}{A(M)}\left\{\frac{A(X)}{L(\Psi)}\Delta X_{10} + \frac{A(U)}{L(\Theta)}\Delta U_{10}\right\},$$

$A(U) = (A(Z) + c)/2 \neq 0$ and $\Delta U_{10} = \Delta Z_{10}/2$.
wherein
Similarly, we derive

$$\Delta N_{10} = \frac{L(\Lambda)}{A(N)}\left\{\frac{A(X)}{L(\Psi)}\Delta X_{10} + \frac{A(V)}{L(\Upsilon)}\Delta V_{10}\right\},$$

wherein $\Lambda_t = (N_t)^2, \Upsilon_t = (V_t)^2, A(V) = (A(Z) - c)/2 \neq 0$, and $\Delta V_{10} = \Delta Z_{10}/2$.
Therefore, our AD is

$$\Delta Y_{10} = \frac{A(X)}{L(\Psi)}\left\{\frac{L(\Gamma)}{A(M)} + \frac{L(\Lambda)}{A(N)}\right\}\Delta X_{10} + \frac{1}{2}\left\{\frac{L(\Gamma)A(U)}{L(\Theta)A(M)} + \frac{L(\Lambda)A(V)}{L(\Upsilon)A(N)}\right\}\Delta Z_{10}. \quad \text{(A3)}$$

Because the contributions of X and Z in Eq. (A3) depend on c, these are indefinite.[2]
If $Z_t > 0$, then $U_t > 0$ and $V_t > 0$. So we can employ the relations (A4) below.
$L(\Psi) = A(X)L(X)$, $L(\Gamma) = A(M)L(M)$, $L(\Lambda) = A(N)L(N)$,
(A4) $(\Theta) = A(U)L(U)$, and $L(\Upsilon) = A(V)L(V)$.

In regard to $L(U)$ and $L(V)$, see also Eq. (20). Substituting the relations (A4) into Eq. (A3) yields

[2] Ang and Liu [2] have shown the method of treatment for $Z_t \geq 0$. It was not based on a theoretical foundation. Their small-value

$$\Delta Y_{10} = \frac{(M) + L(N)}{L(X)} \Delta X_{10} + \frac{1}{2}\left\{\frac{L(M)}{L(U)} + \frac{L(N)}{L(V)}\right\} \Delta Z_{10}. \qquad (A5)$$

Since c is very small, $U_t \approx V_t \approx Z_t/2$ and $M_t \approx N_t \approx Y_t/2$; accordingly, we have $(U) \approx \approx L(V) \quad L(Z)/2$ and $L(M) \approx L(N) \approx L(Y)/2$.

Substituting these relations into Eq. (A5), we obtain the approximation,

$\Delta Y_{10} \approx ((Y)/L(X))\Delta X_{10} + (L(Y)/L(Z))\Delta Z_{10}$. (A6)

Under $Z_t > 0$, we find Eq. (A3) cannot yield Eq. (9) whereas Eq. (A2) can.

The AD of the differential calculus is also easily derived, if the variations of the two independent variables are very small. This AD is

$dY = ZdX + XdZ$.

We must employ in practice the discrete approximation, which is ordinarily given by

$\Delta Y_{10} \approx Z_0 \Delta X_{10} + X_0 \Delta Z_{10}$. (A7)

However this is not correct because the approximation must be symmetric; specifically, the reverse approximation of the AD is

$\Delta Y01 \approx Z1\Delta X01 + X1\Delta Z01$,

which is not equal to the approximation (A7). The proper discrete approximation may be Eq. (7) in this instance.

Table 1: Numerical example 1

	t	Xt	Zt	Yt	$\Delta Yt0$	Decompo sition (A2)		Decompo sition (A3)		Decomp osition (7)	
								c = 0 .0001			
						$\alpha t0\Delta Xt0$	$\beta t0\Delta Zt0$	$\gamma t0\Delta Xt0$	$\delta t0\Delta Zt0$	$A(Zt0)\Delta Xt0$	$A(Xt0)\Delta Zt0$
	0	2.00	0.100	0.20000							
Case 1	1	2.01	-0.010	-0.02010	-0.22010	0.0004777851	-0.2205777851	0.0004777858	-0.2205777858	0.0004500000	-0.2205500000
Case 2	2	2.01	0	0	-0.20000			0.0001445085	-0.2001445085	0.0005000000	-0.2005000000
Case 3	3	2.01	0.101	0.20301	0.00301	0.0010049959	0.0020050041	0.0010049959	0.0020050041	0.0010050000	0.0020050000

Note : For the coefficiets of the variables, see the text.

Table 2: Numerical example 2

Statistics and Mathematical Research Journal
ISSN: 2997-6898 |
Volume 11 Issue 3, July-September, 2023
Journal Homepage: https://ethanpub.online/Journals/index.php/E32
Official Journal of Ethan Publication

	t	X_t	Zt	Yt	$\Delta Yt0$	Decompo sition (A2)		Decompo sition (A3)		Decompo sition (A3)	
								c = 0.01		c = 0 .0001	
						$\alpha t0\Delta Xt0$	$\beta t0\Delta Zt0$	$\gamma t0\Delta Xt0$	$\delta t0\Delta Zt0$	$\gamma t0\Delta Xt0$	$\delta t0\Delta Zt0$
	0	5	2	10							
Case 4	1	7	-1.5	-10.5	-20.5	-141.37441364	120.87441364	-149.94688571	129.44688571	-141.37522151	120.87522151
Case 5	2	7	0	0	-10.0			0.67812295	-10.67812295	0.35170032	-10.35170032
Case 6	3	7	3	21	11.0	4.98855413	6.01144587	4.98855298	6.01144702	4.98855413	6.01144587
Note : For the coefficiets of the variables, see the text.											

To make up the discussions above, we shall explain the computed results for these decompositions using strategy replaces a zero-value with a small positive k and this is very different from our Method 2. Their analytical limit strategy is irrelevant to a log-mean. See also [3].
simple numerical examples. Table 1 shows the results obtained using Eq. (A2), Eq. (A3), and Eq. (7) under the assumption wherein only the variation of X is very small. Conversely, the variation of Z becomes necessarily large when $Z_s > 0$ and $Z_{t\neq s} \le 0$. (Note that Eq. (7) always holds for any real variable.) The two independent variables are given in the table; subscript t (1, 2, or 3) exhibits the comparison period and 0 the base period. The base period is always fixed. For example, $\Delta Y_{t0} = Y_2 - Y_0$, if $t = 2$. The undefined symbols in the table are

$$\alpha_{t0} = \frac{L(\Phi)A(X)}{L(\Psi)A(Y)}, \beta_{t0} = \frac{L(\Phi)A(Z)}{L(\Omega)A(Y)}, \gamma_{t0} = \frac{A(X)}{L(\Psi)}\left\{\frac{L(\Gamma)}{A(M)} + \frac{L(\Lambda)}{A(N)}\right\},$$

$$\delta_{t0} = \frac{1}{2}\left\{\frac{L(\Gamma)A(U)}{L(\Theta)A(M)} + \frac{L(\Lambda)A(V)}{L(\Upsilon)A(N)}\right\}, A(Z_{t0}) = A(Z_t, Z_0), A(X_{t0}) = A(X_t, X_0).$$

Whereas the decomposition of Eq. (A2) cannot apply to Case 2 because of $Z_2 = 0$, Eq. (A3) and Eq. (7) can apply to all cases. (In Table 1, we used c =0.0001 for Eq. (A3).) Comparing the computed results using Eq. (A2) with those from Eq. (A3), we find close relationships for Cases 1 and 3. Comparing the results from Eq. (A3) with those from Eq. (7), we see that the two contributions in Cases 1 and 3 are close. In Case 2, however, the X contribution of Eq.
(A3) is far removed from that of Eq. (7). The reason why this difference is produced is beyond the scope of this paper. (We presently infer that Eq. (7) in Case 2 cannot be used as the discrete approximation to the AD produced by the differential calculus. See also Appendix B.)
Table 2 shows the computed results using Eq. (A2) and Eq. (A3) without the assumption above. For Eq. (A3), we computed two cases : c = 0.01 and c = 0.0001. Two independent variables are also exhibited in the table. Note that the results from Eq. (A2) in Case 6 are equal to those from Eq. (9). The table shows that the two contributions of Eq. (A3) depend on c. Comparing the results using the smaller c value with those using

Statistics and Mathematical Research Journal
ISSN: 2997-6898 |
Volume 11 Issue 3, July-September, 2023
Journal Homepage: https://ethanpub.online/Journals/index.php/E32
Official Journal of Ethan Publication

another value in Cases 4 and 6, we see that the deviations of the former from the results derived using Eq. (A2) are less than those of the latter. Additionally, two X contributions in Case 5 are very different. Recall that a similar behavior is found in Case 2 above.

Appendix B: Comparison of the Difference and Conventional Calculi

We focus on the superior properties of the difference calculus over the conventional calculus using the same examples. Here, only ADs are shown and all variables are positive. **1) Example a:** $Y_t = X_tZ_t$.

The conventional calculus leads to Eq. (7) and the difference calculus to Eq. (9). To compare the former with the latter, we need to employ two approximations explained in Subsection 4.1. These are

$$\frac{G(x)}{A(x)} = \frac{G(x_1, x_0)}{A(x_1, x_0)} = \frac{G(x_1/x_0\,, 1)}{A(x_1/x_0\,, 1)} = \frac{G(1 + (\Delta x_{10}/x_0), 1)}{A(1 + (\Delta x_{10}/x_0), 1)} \approx 1 - \frac{3}{24}\left(\frac{\Delta x_{10}}{x_0}\right)^2$$

$$\frac{G(x)}{L(x)} = \frac{G(x_1, x_0)}{L(x_1, x_0)} \approx 1 - \frac{1}{24}\left(\frac{\Delta x_{10}}{x_0}\right)^2$$

Given that the second-order terms of these approximations are negligible, we have

$A(Z) \approx G(Z) = G(Y)/G(X) \approx L(Y)/L(X)$ and $A(X) \approx G(X) \approx L(Y)/L(Z)$.

Thus, if both X_1/X_0 and Z_1/Z_0 (and thereforeY_1/Y_0) are close to 1, the two components in Eq. (7) approach those in Eq. (9). Below, we shall assume similar approximations as given above.

2) Example b: $Y_t = W_tX_tZ_t$.

For the conventional calculus, we may use transformations of the variables such as $D_t = X_tZ_{t,t} = W_tZ_t$,and $F_t = W_tX_t$. Whenever these are applied to this function, the AD of Eq. (7) derived by the conventional calculus can be utilized. If we utilize D_t, we obtain the AD by repeatedly utilizing Eq. (7):

$\Delta Y_{10} = W_1D_1 - W_0D_0 = A(D)\Delta W_{10} + A(W)\Delta D_{10} = A(XZ)\Delta W_{10} + A(W)(A(Z)\Delta X_{10} + A(X)\Delta Z_{10})$.

Analogously, we have two other ADs using E_t and F_t. We regard the arithmetic mean of these three as the AD for this function derived by the conventional calculus because we employed the arithmetic mean to derive Eq. (7) and Eq. (8). Furthermore, we consider that its three contributions approach those obtained by the difference calculus as shown later. Thus, the AD is

$\Delta Y_{10} = a\Delta W_{10} + b\Delta X_{10} + c\Delta Z_{10}$, (B1)

wherein

$a = (A(XZ) + 2A(X)A(Z))/3$, $b = (A(WZ) + 2A(W)A(Z))/3$,
$c = ((WX) + 2A(W)A(X))/3$.

In sharp contrast, our difference calculus can easily yield this AD, the procedures being

$\Delta \log Y_{10} = \Delta \log W_{10} + \Delta \log X_{10} + \Delta \log Z_{10}$,
$L(Y) \quad L(Y) \quad L(Y)$
$\therefore \Delta Y_{10} = L \_\_\_\_(W)\, \Delta W_{10} + \_\_\_\_(X)\, \Delta X_{10} + \_\_\_\_L(Z)\, \Delta Z_{10}$. (B2)

In applying the approximations stated above to the two ADs of Eq. (B1) and Eq. (B2), we find

$a \approx (G(XZ) + 2G(X)G(Z))/3 = G(XZ) = G(WXZ)/G(W) \approx L(Y)/L(W)$, $b \approx G(WZ) \approx L(Y)/L(X)$, $c \approx G(WX) \approx L(Y)/L(Z)$.

In passing, we note that the differential calculus establishes the following AD: $dY = XZdW + WZdX + WXdZ = (Y/W)dW + (Y/X)dX + (Y/Z)dZ$, which is closely related to Eq. (B2). See also the correspondences between a finite-change variable and an infinitesimal-change variable in Subsubsection 3.1.3.

The AD derived by the conventional calculus has a shortcoming whenever $W_t = X_t$, $X_t = Z_t$, $Z_t = W_t$, or $W_t = X_t = Z_t$. We only show an example wherein we assume $X_t = Z_t$. From Eq. (B1), we obtain

$\Delta Y_{10} = (1/3)\{(A(X^2) + 2(A(X))^2)\Delta W_{10}$

(B3) + $2((WX) + 2A(W)A(X))\Delta Z_{10}\}$.
Since $Y_t = W_t D_t$ and $D_t = (X_t)^2$, we have another AD from the first-mentioned AD,
$\Delta Y_{10} = A(^2X)^2\Delta)W\Delta W_{1010}{+}{+}2A2(AW(W)A)(AX()X\Delta)Z\Delta_{10}X_{10}$. (B4) = $A(X$
Ordinarily, Eq. (B3) does not equal Eq. (B4).
Contrarily, our differential calculus derives the AD from Eq. (B2), giving

$$\Delta Y_{10} = \frac{L(Y)}{L(W)}\Delta W_{10} + 2\frac{L(Y)}{L(X)}\Delta X_{10} = \frac{L(Y)}{L(W)}\Delta W_{10} + 2\frac{L(Y)}{L(X)}\Delta Z_{10}, \qquad \text{(B5)}$$

which is the same as the AD derived using $Y_t = W_t(X_t)^2$. The differential calculus analogously leads to the same AD that is given by: $\mathrm{d}Y = X^2\mathrm{d}W + 2WX\mathrm{d}X = (Y/W)\mathrm{d}W + (2Y/X)\mathrm{d}Z$. (B6)
It is important that both Eq. (B4) and Eq. (B5) always correspond to Eq. (B6) but Eq. (B3) does not. Therefore, the conventional method cannot always produce the ADs that correspond to those derived by the differential one for multiplicative functions composed of three or more independent variables. The same can be found for the next example, if $X_t = Z_t$.
3) Example c: $Y_t = W_t/(X_t Z_t)$,wherein $W_t \neq X_t$ and $W_t \neq Z_t$.
The conventional calculus also exploits the variable transformations, $I_t = X_t Z_{t,t} = W_t/Z_t$, and $K_t = W_t/X_t$.
These lead to
$Y_t = W_t/I_t = J_t/X_t = K_t/Z_t$.
Hence, ADs of Eq. (7) and Eq. (8) can be utilized. We obtain, for example, the following using K_t:

$$\Delta Y_{10} = \frac{K_1}{Z_1} - \frac{K_0}{Z_0} = \frac{K_1 Z_0 - K_0 Z_1}{Z_1 Z_0} = \frac{A(Z)\Delta K_{10} - A(K)\Delta Z_{10}}{Z_1 Z_0},$$

$$\Delta K_{10} = \frac{W_1}{X_1} - \frac{W_0}{X_0} = \frac{A(X)\Delta W_{10} - A(W)\Delta X_{10}}{X_1 X_0}.$$

These two lead to the AD for this function. In this manner, we have three ADs using the three transformations above. The arithmetic mean of these is also regarded as the AD for this function derived by the conventional calculus; specifically,

$$\Delta Y_{10} = \frac{d\Delta W_{10} - e\Delta X_{10} - f\Delta Z_{10}}{X_1 X_0 Z_1 Z_0},$$

where in d, e, and f are

$$d = \big(A(XZ) + 2A(X)A(Z)\big)/3,\ e = \big(Z_1 Z_0 A(W/Z) + 2A(W)A(Z)\big)/3,$$
$$f = \big(X_1 X_0 A(W/X) + 2A(W)A(X)\big)/3.$$

Our difference calculus quickly yields this AD, which is

$$\Delta \log Y_{10} = \Delta \log W_{10} - \Delta \log X_{10} - \Delta \log Z_{10},$$
$$\therefore\ \Delta Y_{10} = \frac{L(Y)}{L(W)}\Delta W_{10} - \frac{L(Y)}{L(X)}\Delta X_{10} - \frac{L(Y)}{L(Z)}\Delta Z_{10}.$$

If we can apply the above-stated approximations to the two ADs, we find the correspondences between the two methods.
The differential calculus leads to the AD:

$$\mathrm{d}Y = \frac{1}{XZ}\mathrm{d}W - \frac{W}{X^2 Z}\mathrm{d}X - \frac{W}{XZ^2}\mathrm{d}Z = \frac{Y}{W}\mathrm{d}W - \frac{Y}{X}\mathrm{d}X - \frac{Y}{Z}\mathrm{d}Z.$$

We can also find a close relation between the ADs derived by the differential and the difference calculi
Now, we say that the conventional calculus is senseless for the functions in Example b and Example c. How about $Y_t = V_t W_t/(X_t Z_t)$? Although our difference calculus can quickly derive this AD, the conventional calculus needs more awkward and complicated procedures than the above. Additionally, the foregoing discussions and those explained in Subsubsection 3.1.3 make clear that the results derived by our

difference calculus are superior to those by the conventional calculus as discrete approximations to those by the differential calculus.

References

Ang, B. W., & Liu, F. L. (2001). A new energy decomposition method: Perfect in decomposition and consistent in aggregation. *Energy*, *26*, 537–548.

Ang, B. W., & Liu, N. (2007a). Handling zero values in the logarithmic mean Divisia index decomposition approach. *Energy Policy*, *35*, 238–246.

Ang, B. W., & Liu, N. (2007b). Negative-value problems of the logarithmic mean Divisia index decomposition approach. *Energy Policy*, *35*, 739–742.

Balk, B. M. (2002–3). Ideal indices and indicators for two or more factors. *Journal of Economics and Social Measurement*, *28*, 203–217.

Balk, B. M. (2008). *Price and quantity index numbers*: *Models for measuring aggregate change and difference*. New York: Cambridge University Press.

Bhatia, R. (2008). The logarithmic mean. *Resonance*, *13*(6), 583–594.

Boole, G. (1958). *Calculus of finite differences* (4th ed.) edited by J. F. Moulton. New York: Chelsea Publishing Company. (Previous editions published under title: *A treatise on the calculus of finite differences*. The first edition was appeared in 1860.)

Carlson, B. C. (1972). The logarithmic means. *American Mathematical Monthly*, *79*(6), 615–618.

Diewert, W. E. (2005). Index number theory using differences rather than ratios. *American Journal of Economics and Sociology*, *64*, 311–360.

Dodd, E. L.(1941). Some generalizations of the logarithmic mean and of similar means of two variates which become indeterminate when the two variates are equal. *Annals of Mathematical Statistics*, *12*, 422–428.

Encyclopedia of Mathematics, "Finite-difference calculus, Computational mathematics, Numerical analysis." [Online] Available: https://www.encyclopediaofmath.org. [Accessed February 13, 2018] [12] Goldberg, S. (1958). *Introduction to difference Equations*. New York: John Wiley & Sons.

Jia, G., & Cao, J. (2003). A new upper bound of the logarithmic mean. *Journal of Inequalities in Pure and Applied Mathematics*, *4*(4), 1–4.

Jordan, C. (1965). *Calculus of finite differences* (3rd ed.). New York: Chelsea Publishing Company.

Paterson, W. R. (1984). A replacement for the logarithmic mean. *Chemical Engineering Science*, *39*(11), 1635–1636.

Pittenger, A. O. (1985). The logarithmic mean in n variables. *American Mathematical Monthly, 92*(2), 99–104.

Spiegel, M. R. (1971). *Theory and problems of calculus of finite differences and difference equations*. New York: McGraw-Hill.

Stolarsky, K. B. (1975). Generalizations of the logarithmic mean. *Mathematics Magazine*, *48*(2), 87–92.

Sugano, H., & Tsuchida, S. (2002). A new approach to discrete data (in Japanese). *Bulletin, Faculty of Business Information Sciences, Jobu University*, *1*, 29–55.

Tsuchida, S. (1997). A family of almost ideal log-change index numbers. *Japanese Economic Review*, *48*, 324–342.

Tsuchida, S. (2014). Interconvertible rules between an aggregative index and a log-change index. *Electronic Journal of Applied Statistical Analysis*, *7* (2), 394–415. [Online] Available: http://siba-ese.unisalento.it/index.php/ejasa/issue/view/1259.

Tsuchida, S. (2015). Strategic decomposition: How to increase total labor productivity. *International Journal of Contemporary Economics and Administrative Sciences*, *5* (3–4), 109–138.

Vartia, Y. O. (1976). Ideal log-change index numbers. *Scandinavian Journal of Statistics*, *3*, 121–126.

Zhao, Y., Liu, Y., Zhang, Z., Wang, S., Li, H., & Ahmad, A. (2017). CO_2 emissions per value added in exports of China: A comparison with USA based on generalized logarithmic mean Divisia index decomposition. *Journal of Cleaner Production*, *144*, 287–298.