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APPLYING LIE SYMMETRY THEORY TO STOCHASTIC DIFFERENTIAL EQUATIONS INFLUENCED BY POISSON PROCESSES

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Abstract:
Lie symmetry theory is a well-established and powerful tool for solving deterministic differential equations, with numerous applications ranging from finding group-invariant solutions to reducing the order of higher-order differential equations and discovering conservation laws. However, its extension to stochastic differential equations (SDEs) is still in its infancy. In contrast to deterministic counterparts, Lie group theory for SDEs remains a relatively unexplored area.
Gaeta and Quintero introduced the first steps towards extending Lie symmetries to stochastic ordinary differential equations (SODEs). They considered a limited class of transformations, known as fiber-preserving transformations. These transformations involve mapping the SODEs from one fiber to another in the manifold, represented as: dX = A(X)dt + B(X)dW, where X is the state variable, A(X) represents the drift term, B(X) is the diffusion term, and dW is the Wiener process. However, it's important to note that this approach is constrained to a specific subset of transformations, capable of preserving the fiber structure. The scope of these transformations is limited compared to the broader universe of possible transformations.
In this manuscript, we aim to further explore the extension of Lie symmetries to SDEs, moving beyond the restrictions of fiber-preserving transformations. We strive to enhance our understanding of the relationship between symmetries in stochastic systems and their corresponding Fokker-Planck equations. This work contributes to the ongoing development of Lie symmetry theory in the realm of stochastic differential equations, offering new avenues for addressing complex stochastic systems.

Keywords: Lie symmetry theory, stochastic differential equations, fiber-preserving transformations, Fokker-Planck equation, and symmetries

1. Introduction

Lie symmetry theory of deterministic differential equations is well understood in literature [16, 17, 18, 19, 20] and can used for many important applications in the context of differential equations. For instance, for determination of group-invariant solutions, solving the first order differential equation, reducing the order of higher ODE, reducing the number of variables of partial differential equations and finding conservation laws.

In contrast to the deterministic differential equation, only a few attempts have been made to extend Lie group theory to the stochastic differential equation. It is worth noticing that the theory is still developing. Gaeta and Quintero [6] made the first approach to extend Lie symmetry of differential equations to It stochastic ordinary differential equations by which they consider a small class of transformations, i.e., fiber preserving transformations

$$\frac{\ }{\ } = \ (, \quad , \quad), \qquad = \ (, \quad).$$

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The method has been used to study the relationship between symmetries of stochastic systems to the symmetries of their corresponding Fokker-Planck equation. This is a restricted transformation that can only work to a fiber-preserving class of transformations which is a small sub-class of all possible transformations.
The second attempted [3, 4, 5, 8, 10, 15] succeed in applying symmetry transformations that include all the dependent variables in the transformation

_ = (, ,), = (, ,).

This approach has been used to study the symmetry of scalar stochastic ordinary differential equations of first order [4] which reconciled the works of Meleshko S. V., Srihirun B. S. and Schultz E. [8] and Wafo Soh and F.M. Mahomed [10]. Furthermore, the formal method for finding Lie Point symmetries of scalar It stochastic differential equations of the first order driven by the Wiener process was also discussed by E. Fredericks [3] with intention to correct and reconcile the finding of Srihirun and Schultz [8].
To the best of our knowledge in literature, all the methods above were applied only to the It stochastic differential equations driven by Wiener processes [3-12]. In this paper we extend the Lie symmetry methods to the class of It stochastic differential equations driven by a Poisson process by implementing a more generalized It formula and following the methodology of G. Gaeta [6] and E Fredericks and F. M. Mahomed [3].
We consider an It stochastic differential equation driven by Poisson processes;

( ) = , ( ) + , ( ) ( ) (1.1)

with initial condition (0) = . So, equation (1.1) can be written in integral form as

( ) = + , ( ) + , ( ) ( ). (1.2)

Where , ( ) and , ( ) are × 1 dimensional drift vector coefficients and Poisson diffusion coefficient respectively, which are assumed to satisfy Ikeda and Watanabe conditions for the uniqueness and existence of the solution of (1.1)while ( ) is the infinitesimal increment of the Poisson Process [12, 13, 14].
Symmetries of (1.1)are analysed by considering an infinitesimal generator

= (,) + (,) . (1.3)

The determining equations for It stochastic differential equations (SDE) driven by Poisson processes (1.1) are derived using It calculus and are found to be non-stochastic.
Starting with an arbitrary function , ( ) which is once differential with respect to the spatial coordinate and differentiable once with respect to temporal variable , the It Poisson diffusion process for , ( ) of (1.1) exists [1, 2] and is

, ( ) = + + (, ( ) + , ( ) − , ( ) ( ). (1.4)

The Einstein summation convention is assumed throughout this paper. Let

$\Gamma_{( )}$, ( ) = + (1.5)

and

$\Gamma^*_{( )}$, ( ) = (, ( ) + , ( ) − , ( ) (1.6)

Therefore (1.4) can be written as;

, ( ) = Γ( ) , ( ) + Γ(*) , ( ) ( ). (1.7)
Using the It multiplication properties of Poisson processes [1, 2]

( ) · ( ) = ( ), ( ) · = 0 and · = 0

And application of infinitesimal transformations the determining equations for (SDE) with Poisson processes
are derived and are non-stochastic. The main result can be summarised as

Theorem 1.1: The It stochastic differential equation driven by Poisson processes

( ) = , ( ) + , ( ) ( ) (1.8)
Where , ( ) and , ( ) are the × 1 dimensional drift vector coefficient and the Poisson diffusion coefficient, with infinitesimal generator

= (,) — + (,) — (1.9)
Has the following determining equations;
Γ( ) + —Γ( ) + −Γ( ) , ( ) = 0, (1.10) 2
—Γ ( ) + −Γ(*) , ( ) = 0 (1.11)
2
with additional conditions,

Γ(*) , ( ) = 0, Γ( ) , ( ) = . (1.12)
Where the operators Γ , ( ) and Γ* , ( ) are defined as in (1.5) and (1.6), and > 0 is called the intensity of the jump process or jump rate.

1. Lie Group Transformation

Consider a one parameter group of transformations of the time index and the spatial variable respectively

= (, ,), — = (, ,)
with the infinitesimals

— = (,), — = (,)

Satisfying the following initial conditions at = 0

= , —| = .
A one parameter Lie group of infinitesimal transformations is therefore

= + (,) + ( ) (2.1)
And

( ) = ( ) + (,) + ( ) (2.2)

Where is the parameter of the group, with the corresponding generator of the Lie algebra of the form

= (,) — + (,) —.

The differential point transformations of the spatial, temporal and the Poisson process variables are as follows

= + (,) + ( ), (2.3)

( ) = ( ) + (,) + ( ) (2.4)

and

= ( ) + — + ( ) + ( ). (2.5) 2

Using the It formula(1.7), we have the spatial and temporal infinitesimals in It forms as

$= \Gamma_{( )} + \Gamma^{*}_{( )} ( )$ (2.6)

$= \Gamma_{( )} + \Gamma^{*}_{( )} ( )$ (2.7)

where $\Gamma_{( )}$, $\Gamma_{(*)}$, $\Gamma_{( )}$ $\Gamma_{(*)}$are the drift and diffusion coefficients of the spatial and temporal infinitesimals, respectively defined using the operators (1.5) and (1.6).

By substitution of the infinitesimal of spatial (2.6) and temporal variables (2.7) in (2.3), (2.4) and (2.5), and also using the It multiplication properties we proceed to get the group transformations of temporal, spatial and jump variables in It forms

$= + \Gamma_{( )} + \Gamma^{*}_{( )} ( ) + ( ),$ (2.8)

$= + \Gamma_{( )} + \Gamma_{(*)} ( ) + ( )$ (2.9)

and

$= ( ) + \frac{\Gamma( ) + \Gamma(*) ( )}{2} + ( ) + ( ).$ (2.10)

Expending the It infinitesimal of the jump variable (2.10) by utilising the Poisson process differential multiplication properties we get

$= ( ) + \frac{}{2} \Gamma_{( )} + \Gamma^{*}_{( )} ( ) + \Gamma_{( )} ( ) + \Gamma^{*}_{( )} \frac{( )}{} + ( ).$ (2.11)

1.1 Invariance form of the Spatial Process

To ensures the recovery of the finite transformations from the infinitesimal transformation, we need to transform ( ) into

( ) = , ( ) ( ) + , ( ) (2.12)

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where the transformed drift component , ( ) and jump component , ( ) using the infinitesimal generator

( ) ( )

= , + , ,
respectively are

, ( ) = + , ( )

= , ( ) + + , ( ) (2.13)

and

, ( ) = + , ( )

= , ( ) + + , ( ) (2.14)

1.2 Poisson Invariance Properties

We apply the invariance to the moments of the Poisson process to ensure it remains invariant under the group transformations, *viz* the instantaneous mean and variance of the Poisson process which are:

[( )] = · (2.15)

[( ) · ( )] = · . (2.16)

The invariance of the instantaneous mean of the transformed Poisson process under new measure is

( ) = · (2.17)

Expanding (2.17) using the It forms of jump (2.8) and temporal group transformations (2.11) we get

$\Gamma^{*}_{( )}$, ( ) = 0 (2.18)

Next, we apply the invariance form to instantaneous variance of the transformed Poisson process measure (2.16) from which using (2.11) we have

( ) · ( ) = · (2.19)

Thus, using (2.18) and the It temporal group transformation (2.8) we have derived the following generalized random time change formula

= $\Gamma_{( )}$( )(2.20)

With

$\Gamma_{( )}$, ( ) = = (2.21)

Using the probabilistic invariance property of the transformed time index differential, i.e.,

= . (2.22)

Finally, we can conclude from (2.18) the temporal infinitesimal (,)does not depend on , therefore (,) = ( ).

Definition 2.1 The infinitesimal transformations (2.3) and (2.4) i.e.

= + (,) + ( ), = + (,) + ( )(2.23)

Are called Lie symmetry transformations of (1.1) if they leave the It stochastic differential equation (1.1)

( ) = , ( ) + , ( ) ( )(2.24)

And the infinitesimal moments for the differential Poisson process i.e.,

[( )] = · , [( ) · ( )] = · [ ] = (2.25)

Invariant. Where > 0 and is the jump intensity and group parameter respectively.

2. Determining Equations

In this section, we will derive the determining equations for the admitted symmetries of (1.1). The intention is to transform ( ) into

( ) = , ( ) + , ( ) (3.1)

Substituting the transformed drift coefficient (2.13), Poisson vector coefficients(2.14), It forms of temporal (2.8) and Poisson group transformation (2.11) into (3.1) we get

$$= ( ) + \Gamma_{( )} , ( ) + \frac{}{2}\Gamma_{( )} , ( ) +$$

$$+ \Gamma^{*}_{( )} , ( ) + \frac{}{2}\Gamma_{( )} , ( ) + ( ) \quad (3.2)$$

Therefore, by comparing transformed stochastic differential equation (3.2) and the It form of the spatial group transformation (2.9) we have the following determining equations

$$\Gamma_{()} + \frac{}{2}\Gamma_{()} + -\Gamma_{()} , ( ) = 0, \quad (3.3)$$

and

$$\frac{}{2}\Gamma_{( )} + -\Gamma_{(*)} , ( ) = 0. \quad (3.4)$$

The invariance of the instantaneous mean of the transformed differential Poisson process (2.17) gives additional conditions i.e., from (2.18) and (2.21)we get

$\Gamma^{*}_{(\ )}$, ( ) = 0 $\Gamma_{(\ )}$, ( ) = . (3.5)

Equation (3.3) can be interpreted using the definition of first prolongation of an infinitesimal generator for non-stochastic ordinary differential equations as follows

[ ] = + [ ] ___. (3.6)

Where
= ___ = (3.7) and

[ ]
= ( ) − ( ) (3.8)

= + — + — — (3.9)

with total time derivative defined as
= + + — + ⋯ — (3.10)

Using the definition of first prolongation on − at = , can be expressed as
[ ](−) = [ ] − ( ). (3.11)
Using (3.8) and (3.11) equation (3.4) can be written as
[ ](−) − — $\Gamma_{(\ )}$ — + — = 0. (3.12)
2
Where the operators $\Gamma_{(\ )}$, ( ) , $\Gamma_{(*)}$, ( ) are defined in (1.5), (1.6) respectively, and is called the jump rate or jump intensity of the Poisson process.
Remark 3.1: The extra condition obtained from the invariance of the instantaneous mean of the transformed differential Poisson process (2.17) forces the temporal infinitesimal (,) to be a function of the time variable only. This implies that we are now dealing with a fiber-preserving infinitesimal generator i.e.,

= ( ) — + (,) ___. (3.13)

3. Applications

In this section, we are going to apply the derived determining equations of Poisson It stochastic differential equations obtained in the previous section to some Poisson process models to show how the determining equations can be used to find the admitted Lie point symmetries of each model.
Example 4.1: Consider the Poisson SDE, linear in the state process ( ), with constant coefficients,
( ) = ( ) ( ) + ( ) ( ) (4.1)
With initial condition ( ) = > 0, ( ) = 2 is called the drift or deterministic coefficient and ( ) = 1 is the jump amplitude coefficient of the jump term, with jump rate = .
Using the determining equations (3.3) (3.4) respectively we have

$$2\ \Gamma_{(\ )} + \frac{\Gamma_{(\ )}}{2} + 2\ (\ ,\ ) - \Gamma_{(\ )}\quad ,\ (\ )\ = 0 \tag{4.2}$$

$$2\ \frac{(\ )}{} + \frac{}{2}\frac{(\ )}{} + 2\ (\ ,\ ) - \frac{(\ ,\ )}{} - 2\ \frac{(\ ,\ )}{} = 0 \tag{4.3}$$

and

$$\frac{\Gamma_{(\ )}}{2} + (\ ,\ ) - \Gamma^{*}_{(\ )}\quad ,\ (\ )\ = 0 \tag{4.4}$$

$$\frac{}{2}\frac{(\ )}{} + (\ ,\ \ +\ ) + (\ ,\ ) = 0. \tag{4.5}$$

Using (2.18) and (2.21) we get the temporal infinitesimal as

( ) = + . (4.6)

Substituting the temporal infinitesimal (4.6) in (4.3) and (4.5) respectively gives

$$\frac{(\ \ +4)}{2} + 2\ (\ ,\ ) - \frac{(\ ,\ )}{} - 2\ \frac{(\ ,\ )}{} = 0 (4.7)$$

and

$$\frac{}{2} + 2\,(\ ,\ \ ) - (\ ,2\ ) = 0. (4.8)$$

Differentiating (4.7) with respect to gives

$$\frac{(\ \ +4)}{2} + 2\frac{(\ ,\ )}{} - \frac{(\ ,\ )}{} - 2\frac{(\ ,\ )}{} - 2\ \frac{(\ ,\ )}{} = 0. (4.9)$$

Differentiating (4.8) with respect to gives

$$\frac{}{2} + 2\frac{(\ ,\ )}{} - 2\frac{(\ ,2\ )}{} = 0. \tag{4.10}$$

Differentiating (4.10) with respect to gives

$$\frac{(\ ,\ )}{} = \frac{(\ ,2\ )}{}. \tag{4.11}$$

Equation (4.11) implies

$$\frac{(\ ,\ )}{} = \frac{(\ )}{} = \frac{(\ )}{}. \tag{4.12}$$

Solving the differential equation (4.12) we get

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$(, \quad) = \quad () \quad + \quad ().$ (4.13)

By substituting (4.13) into (4.9) we get

$$\frac{(\quad + 4)}{2} = \frac{()}{} + 2 \frac{()}{}. \quad (4.14)$$

When differentiating (4.14) with respect to time we get

$$\frac{()}{} = 0. \quad (4.15)$$

Solving the ordinary differential equation (4.15) implies ( ) is linear in i.e.,

$() = \quad + \quad .$ (4.16)

After substituting (4.16) into (4.13) we arrive at the spatial infinitesimal

$(, \quad) = (\quad + \quad) \quad + \quad ().$ (4.17)

Substituting (4.17) into (4.14) results in

$$\frac{(\quad + 4)}{2} = \quad + 2 \frac{()}{}, \quad (4.18)$$

Which implies that

$$\frac{()}{} = \frac{\frac{(\quad)}{} -}{2}. \quad (4.19)$$

Solving the differential equation (4.19) for ( ) finally gives

$$() = \frac{\frac{(\quad)}{} -}{2}(\quad | \, | - \quad) + \quad + \quad , \quad (4.20)$$

Therefore, using (4.20) the special infinitesimal (4.17) can be written as

$$(, \;) = (\quad + \quad) \quad + \frac{\frac{(\quad)}{} -}{2}(\quad | \, | - \quad) + \quad + \quad . \quad (4.21)$$

However, substituting (4.21) in (4.8) we have

$$\frac{}{2} + 2 \quad (\quad + \quad) \quad + \frac{\frac{(\quad)}{} -}{2}(\quad | \, | - \quad) + \quad +$$

$$= 2(\quad + \quad) \quad + \frac{\frac{(\quad)}{} -}{2}(2 \quad |2 \; | - 2 \;) + 2 \quad + \quad . \quad (4.22)$$

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Which can be simplified to get

$$\frac{}{2} + \quad = \frac{\frac{(\quad)}{} - }{2}(\quad |4|). \qquad (4.23)$$

Further comparison of the coefficients of powers of in (4.23), gives

- $: \quad = \frac{(\quad)}{} - \frac{}{|\ |}$
- $: = 0.$

Thus, the spatial infinitesimal (4.21) finally becomes

$$(\ ,\quad) = \frac{|4|(\quad + 4) - 2}{|16|} + \frac{|\ | -}{|16|} + \quad + \quad . \qquad (4.24)$$

So we have three symmetry generators corresponding to the infinitesimals

$$= \ — + \frac{|4|(\quad + 4) - 2}{|16|} + \frac{|\ | -}{|16|} —, \qquad = —, \qquad = 2 \ —. \quad (4.25)$$

The infinitesimal generators (4.25) give the following Lie bracket relations in *Table 1* below

,			
	0	–	$-\frac{}{\lvert 16 \rvert}$
		0	0
	$\frac{}{\lvert 16 \rvert}$	0	0

Table 1: Commentator table for the Lie algebra generators (4.25)

The commentator table shows that the infinitesimals generators (4.25) is closed under Lie bracket relations and hence is a Lie algebra, where is linear combination of given as

$$= \quad + \quad h \quad = \frac{|16| - 1 + \quad |2|}{|16|}. \qquad (4.26)$$

Example 4.2: Consider a Poisson driven stochastic differential equation

$$(\ ) = - \ + \qquad (\ ) \ \ h \neq 0 \qquad (4.27)$$

And initial condition (0) = .

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Using the determining equations (3.3) (3.4)respectively we have

$$— \quad — – \quad — + \frac{}{2} \quad — – \quad — – 2 \quad (,) = — – \quad — \tag{4.28}$$

and

$$\frac{}{2} \quad — – \quad — = (, +) – (,). \tag{4.29}$$

Using equation (2.18) and (2.21) we get the temporal infinitesimal as

$$( ) = \quad + \quad . \tag{4.30}$$

Using temporal infinitesimal (4.30) in (4.28) and (4.29) we respectively have

$$\frac{}{2} – \quad – 2 \quad (\quad + \quad) = — – — \tag{4.31}$$

And

$$(, +) – (,) = \frac{}{2}. \tag{4.32}$$

Differentiating (4.31) and (4.32) with respect to respectively gives

$$— – — = 0 \tag{4.33}$$

and

$$\frac{(, +)}{} = \frac{(,)}{}. \tag{4.34}$$

Equation (4.34) implies

$$\frac{(,)}{} = \frac{( )}{} = ( ). \tag{4.35}$$

Differentiating (4.35) with respect to gives

$$— = 0, \tag{4.36}$$

Solving the differential equation (4.36) we have

$$(, \quad) = \quad ( ) \quad + \quad ( ). \tag{4.37}$$

Substituting (4.37) into (4.33) implies

$$\frac{( )}{} = 0. \tag{4.38}$$

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Equation (4.38) implies ( ) is constant i.e.,

( ) = , (4.39)

Therefore, using (4.39) and (4.37) we have

(,) = + ( ). (4.40)

Substituting (4.40) into (4.32) gives this relation

= 2 . (4.41)

Using (4.40) and (4.41), equation (4.31) gives

$$\frac{}{2} - 3 - 2 \quad = \frac{(\ )}{} - \frac{}{2}. \qquad (4.42)$$

Solving the differential equation (4.42) gives

$$(\ ) = \quad \frac{}{2} - \frac{5}{6} \quad - \quad + \quad . \qquad (4.43)$$

Therefore, substituting (4.43) into (4.40) the spatial infinitesimal finally becomes

$$(\ ,\ ) = \quad \frac{}{2} - \frac{5}{6} + \frac{}{2} \quad -+ \quad . \qquad (4.44)$$

Finally, the Poisson diffusion model admitted three dimensional symmetry infinitesimal generators;

$$= \quad - + \quad \frac{}{2} - \frac{5}{6} + \frac{}{2} \quad -, \qquad = - - \quad -, \qquad = -. \qquad (4.45)$$

With the corresponding Lie bracket relations of the generators (4.45) given in *Table 2* as

,			
	0	$-$	$-\frac{}{2}$
		0	0
	$\frac{}{2}$	0	0

Table 2: Commentator table for the Lie algebra generators (4.45)

The Lie bracket relations in *Table 2* above show that the infinitesimal generator (4.45) satisfied Lie commutative relation properties and hence forms a Lie algebra, where

= $-\frac{}{}$ is the linear combination of .

4. Conclusion

Lie Symmetry analysis for It stochastic differential equations driven the by Poisson processes was carried out, infinitesimals of the Poisson process ( ) were derived using the moments invariance properties of the process. Determining equations were derived and found to be deterministic even though they describe stochastic differential equation. Examples are given to show how the determining equations can be used to find the symmetries, symmetries admitted by (1.1) are found to be fiber-preserving symmetries. Finally, the Lie bracket relation was obtained which shows that all the infinitesimal generators found are closed under the Lie bracket and hence they form a Lie algebra. Classification of the given examples is presented in *Table 3*.

Group Dimension	Basis Operators	Equations
3	$= \frac{}{} + \frac{\|4\|(\ + 4) - 2}{\|16\|} + \frac{\| \ \| -}{\|16\|} \frac{}{},$ $= \frac{}{},\quad = 2 \frac{}{}.$	( ) = ( ) 2 + ( )
3	, = $= \frac{}{} + \frac{}{2} - \frac{5}{6} + \frac{}{2} \frac{}{}$ = $\frac{}{} - \frac{}{},\quad \frac{}{}$	( ) = − + ( ), ≠ 0

Table 3: Lie Group Classification

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