

INTEGRO-DIFFERENTIAL EQUATIONS-BASED MULTISCALE WAVELET SHRINKAGE AND MULTI-DIMENSIONAL CONTINUOUS SHEARLET TRANSFORM

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Abstract:

In recent years, the study of wavelets has gained substantial attention, revealing their profound relevance in various branches of pure and applied mathematics. Wavelet theory has proven to be a valuable tool across multiple scientific disciplines, engineering, and other fields. Notably, wavelets have played a pivotal role in developing essential technologies such as the FBI's digital fingerprint image compression standard in 1993 and JPEG2000, the current industry standard for image compression. They have facilitated the analysis of fundamental linear operators within harmonic analysis and partial differential equations, as demonstrated in the Calderon-Zygmund theory.

In the realm of image processing, wavelet shrinkage emerges as a crucial technique to address core issues, including image compression, noise reduction, feature extraction, and object recognition. One fascinating aspect of wavelet shrinkage is its diverse mathematical foundations. It finds inspiration in various mathematical domains, encompassing partial differential equations, the calculus of variations, harmonic analysis, and statistics. This interdisciplinary nature has invigorated the application of wavelets in the realm of engineering and technology.

Moreover, pioneering work by Donoho and others has delved into wavelet shrinkage methods within the context of minimax estimation. Their investigations have revealed that wavelet shrinkage provides asymptotically optimal estimates for noisy data, surpassing the performance of linear estimators. This opens new horizons for utilizing wavelets in diverse engineering applications.

Keywords: Wavelets, Image Processing, Wavelet Shrinkage, Engineering Applications, Minimax Estimation.

1. Introduction

During the last years, there has been growing interest in studying the wavelets and found that wavelet theory is important on several areas of pure and applied mathematics. It gave the understanding of many problems in various sciences, engineering and other disciplines and it includes, among its notable successes, the wavelet base digital fingerprint image compression standard adopted by the FBI in 1993 and JPEG2000, the current standard for image compression. It provided a very suitable vehicle for the analysis of the core linear operators of harmonic analysis and partial differential equations (Calderon – Zygmund theory). Given an image, the generic problems of image processing are compression, noise reduction, feature extraction and object recognition of which wavelet shrinkage plays an important role. It should be noted that a very interesting thing about wavelet shrinkage ([1], [7-9], [11-12], [17], [19]) is that it can be motivated from very different fields of mathematics, namely partial differential equations, the calculus of variations, harmonic analysis or statistics, which, conversely gives a great impetus to the applications of wavelet to many engineering fields. Of these, Donoho et al. analyzed wavelet shrinkage

methods in the context of minimax estimation and showed that wavelet shrinkage generates asymptotically optimal estimates or noisy data that outperform any linear estimator [7-9].

It is notable that the continuous wavelet transform has many advantage over the discrete wavelet transform, such as the weaker limited condition for the generate function, the arbitrary choice of the scale and translation, especially the wide application in the fields of pattern recognition, feature extraction and detection.

Using the idea of understanding wavelets as smoothed derivative operators, Didas S and Weickert J [13] describe one dimensional wavelet shrinkage as approximation to a novel integro-differential evolution equation and compare the corresponding energy functional that uses both smoothed derivative operators within the penalties and integration over all scales, with the classical regularization ones. In [16] Liu, X. Feng and Min Li extend the results of [13] to d -dimensional ($d \geq 2$) case. In one dimension, there are wavelets not arising from an MRA. Also, wavelets can be defined by replacing dyadic dilations with dilations by $\lambda > 1$, where need not be an integer. In the situation of non-dyadic dilations, the construction of the orthonormal bases associated with the wavelet may require more than one generator, namely, if $\lambda = \frac{p}{q} > 1$ and p, q are relatively prime, then q generators are needed. In order to avoid multiple wavelet generators in higher dimension we restrict our attention to $d \times d$ integer matrices with $|\det| = 2$ where the dilation set $2 : \mathbb{R}$ is replaced by $\lambda : \mathbb{R}$. The d -dimensional continuous wavelet transform used in [16] is able to describe the local regularity of functions and distribution and detect the location of singularity points through it decay at fine scales, it does not provide additional information about geometry of the set of singularities. Several constructions have been introduced, starting with the wedgelets [6] and ridgelets [3]. Among the most successful constructions proposed in the literature, the curvelets [4] and shearlets [14] achieve this additional flexibility by defining a collection of analyzing functions ranging not only over various scales and locations, like traditional wavelets, but also over various orientations and with highly anisotropic supports.

All of these motivate us to extend the results of [16]. In this paper we have considered wavelet transform associated with more general dilation groups. This generalized transform is called the continuous shearlet transform. Firstly, we relate a d -dimensional continuous mother wavelet (ψ) with a fast decay and vanishing moments to the sum of the order partial derivative of a group function $(\phi)(| \lambda | = 1)$ with fast decay, which also makes shearlet transform equal to the sum of the smoothed partial derivative operators. Finally, d -dimensional continuous shearlet transform can be explained as a weighted average of pseudo-differential equations.

2. Notations and d -Dimensional Fourier Transform with Some Properties

We write $\mathbf{x} = (x_1, \dots, x_d)$, $\mathbf{y} = (y_1, \dots, y_d) \in \mathbb{R}^d$, $\mathbb{R}^d = \cup \{0\}$, let $|\mathbf{x}| = \sum_{i=1}^d |x_i|$, $|\mathbf{y}| = \sum_{i=1}^d |y_i|$, $\mathbf{z} = \mathbf{x} + \mathbf{y}$. For $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$, $\mathbf{y} = (y_1, \dots, y_d) \in \mathbb{R}^d$ and the partial derivative operators be $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d} = \dots$.

For $1 \leq p < \infty$, $\mathbf{x} \in \mathbb{R}^d$, let $(f) = f(\mathbf{x}) : \mathbb{R}^d \rightarrow \mathbb{R}$; $\int_{\mathbb{R}^d} |f(\mathbf{x})| dx < \infty$.

The space $L^p(\mathbb{R}^d)$ is the Hilbert space of all square (Lebesgue) integrable functions endowed with inner product $\langle f, g \rangle = \int_{\mathbb{R}^d} f(\mathbf{x}) \overline{g(\mathbf{x})} dx$. The Fourier transform is the unitary operator that maps $f \in L^2(\mathbb{R}^d)$ into the function defined by $(\hat{f})(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} f(\mathbf{x}) e^{-i \boldsymbol{\xi} \cdot \mathbf{x}} dx$.

when $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and by the appropriate limit for the general $f \in L^2(\mathbb{R}^d)$. The function $\hat{f}$ is also square integrable. Indeed, Fourier transforms maps (f) one-to-one onto itself and the inverse Fourier transform is defined by $(f) = \int_{\mathbb{R}^d} \hat{f}(\boldsymbol{\xi}) e^{i \boldsymbol{\xi} \cdot \mathbf{x}} d\boldsymbol{\xi}$.

The convolution of two functions $f, g \in L^1(\mathbb{R}^d)$ is defined as for all $\mathbf{x} \in \mathbb{R}^d$

$$(f * g)(\mathbf{x}) = \int_{\mathbb{R}^d} f(\mathbf{y}) g(\mathbf{x} - \mathbf{y}) d\mathbf{y}.$$

The following properties of Fourier transform will be used throughout the paper.

1. Convolution. For $f, g \in L^1(\mathbb{R}^n)$, we have

$$(f * g)(\xi) = \hat{f}(\xi) \hat{g}(\xi).$$

2. Time partial derivatives. For $f \in L^1(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}^n$, we have

$$(\partial^\alpha f)(\xi) = (-i)^\alpha \hat{f}(\xi).$$

3. Frequency partial derivatives. For $f \in L^1(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}^n$, $1 \leq \alpha_j \leq \infty$, we have

$$\widehat{\partial^\alpha f}(\xi) = (i)^\alpha \hat{f}(\xi).$$

4. Scaling and translation property. For $f \in L^1(\mathbb{R}^n)$, $a > 0$ and $b \in \mathbb{R}^n$, we have

$$\widehat{f(ax + b)}(\xi) = \exp(-ib \cdot \xi) \hat{f}(a\xi).$$

3. Continuous Wavelets on $\mathbb{R}^n$

In this section, we will study the continuous wavelets and continuous wavelet transform on $\mathbb{R}^n$. Many of the concepts of one dimension can be extended to n -dimensions ($n \in \mathbb{N}$, $n > 1$) with translations replaced by translations and the dilation set $\{2^j : j \in \mathbb{Z}\}$ replaced by $\{a^j : j \in \mathbb{Z}\}$, where a is an $n \times n$ real matrix, each of whose eigenvalues has magnitude larger than one. Like the Fourier transform, the wavelet transform has a discrete and a continuous version. For the continuous wavelet transform, one has weaker conditions, especially the orthogonality not necessary for an invertible continuous wavelet transform. So, it is convenient to focus our attention on Parseval

2-frame (rather than orthonormal wavelets). Thus, given the matrix $a = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$, for $n = 2$ and

$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ the wavelet system

$$\psi_j(x) = \hat{\psi}(a^{-j}x) \quad (3.1)$$

$$= |\det a|^{-j} \hat{\psi}(x), \quad x \in \mathbb{R}^n,$$

is a Parseval frame. That is,

$$\langle \psi_j, \psi_k \rangle = \delta_{j,k}, \quad \|\psi_j\| = 1 \quad (j \in \mathbb{Z})$$

for all $j \in \mathbb{Z}$.

In general, the construction of two-dimensional wavelets is significantly more complicated than the one-dimensional case. In particular, it is not known whether there exists continuous compactly supported orthonormal wavelet analogous of the one-dimensional Daubechies wavelets associated with the dilation quincunx matrix $a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

. However, it turns out that a rather simple change in the definition of the dilation (3.1) produces much

1 simpler constructions of Haar type wavelets in dimension two.

In [15] K. Guo, W-Q. Lim, D. Labate, G. Weiss and E. Wilson, introduced the notion of wavelet system with composite dilations, which are the systems of the form

$\{ \quad : \in \mathbb{R}, \in \mathbb{R}, \in \mathbb{R}, \in \mathbb{R} \}$, where $\mathbb{R}$ is a group of matrices with determinant of absolute value 1 and $\mathbb{R}$ is a group of expanding matrices, in the sense that all eigenvalues have magnitude larger than one. Many different groups of dilations have been considered in the literature, such as the crystallographic and shear group, which are associated with very different properties. One special benefit of this framework is the ability to produce wavelet like systems with geometric properties going far beyond traditional wavelets. For example, one can construct waveforms whose supports are highly anisotropic and that range not only over various orientations, making these functions particularly useful in image processing applications.

Continuous Wavelet Transform: Let the general linear group $(\mathbb{R}, \mathbb{R})$ of $\times$ invertible real matrices acts on $\mathbb{R}$ by linear transformations. The semidirect product of $(\mathbb{R}, \mathbb{R})$ and $\mathbb{R}$ is called the general affine group on $\mathbb{R}$, since each invertible affine map on $\mathbb{R}$ has the form $(\mathbb{R}, \mathbb{R}) \cdot \mathbb{R} = (\mathbb{R}, \mathbb{R}) + \mathbb{R}$ for a unique $(\mathbb{R}, \mathbb{R}) \in (\mathbb{R}, \mathbb{R})$. Thus, the group law $(\mathbb{R}, \mathbb{R}) \cdot (\mathbb{R}, \mathbb{R}) = (\mathbb{R}, \mathbb{R}) + (\mathbb{R}, \mathbb{R})$ for $\mathbb{R}$ corresponds to the composition of the associated affine maps and $(\mathbb{R}, \mathbb{R}) = (\mathbb{R}, -)$. A unitary representation of $(\mathbb{R}, \mathbb{R})$ acting on $(\mathbb{R})$ is defined by

$$(\mathbb{R}, \mathbb{R}) \cdot (\mathbb{R}) = |\det \mathbb{R}| \cdot [(\mathbb{R}, \mathbb{R}) \cdot \mathbb{R}]$$

$$= |\det \mathbb{R}| \cdot [(\mathbb{R}, -) \cdot \mathbb{R}] = \mathbb{R}(\mathbb{R}),$$

and

$$(\mathbb{R}, \mathbb{R}) \cdot (\mathbb{R}) = |\det \mathbb{R}| \cdot [(\mathbb{R}, *) \cdot \mathbb{R}].$$

For $\mathbb{R} \in (\mathbb{R})$, the continuous wavelet transform we associated with $\mathbb{R}$ and $\mathbb{R}$ is defined by

$$(\mathbb{R}, \mathbb{R}) = \langle \mathbb{R}, \mathbb{R} \rangle \\ = |\det \mathbb{R}| \cdot \int (\mathbb{R}) \cdot (\mathbb{R}, -) \cdot \mathbb{R},$$

and it maps $\mathbb{R} \in (\mathbb{R})$ into a space of functions on $\mathbb{R}$. For $\mathbb{R}$ a closed subgroup of $(\mathbb{R}, \mathbb{R})$, $\mathbb{R} =$

$\{(\mathbb{R}, \mathbb{R}) : \mathbb{R} \in \mathbb{R}, \mathbb{R} \in \mathbb{R}\}$ is a closed subgroup of $(\mathbb{R}, \mathbb{R})$ and the left Haar measures on $\mathbb{R}$ are the product measures $(\mathbb{R}, \mathbb{R}) = (\mathbb{R})$, where $\mathbb{R}$ is a left Haar measure on $\mathbb{R}$. It should be mentioned that, we seek conditions on $\mathbb{R}$ and $\mathbb{R}$ for which restricting to $\mathbb{R}$ gives an isometry from $(\mathbb{R})$ into $(\mathbb{R}, \mathbb{R})$ and we have

$$= \langle \mathbb{R}, \mathbb{R} \rangle \cdot (\mathbb{R}, \mathbb{R}) \cdot (\mathbb{R}),$$

for each $\mathbb{R} \in (\mathbb{R})$. As given in [2], this holds if and only if

$$\int \mathbb{R} \cdot (\mathbb{R}) = 1, \text{ for } \mathbb{R} \in \mathbb{R}, \quad (3.2)$$

in which case $\mathbb{R}$ is a continuous wavelet with respect to $\mathbb{R}$. Particularly if $\mathbb{R} = \{ \mathbb{R} : \mathbb{R} > 0 \}$, where $\mathbb{R}$ is the

$\times$ identity matrix and $(\mathbb{R}) = \mathbb{R}$, the condition (3.2) reduces to the classical Colderon condition

$$\int |\mathbb{R}(\mathbb{R})| \cdot \mathbb{R} = 1, \text{ for } \mathbb{R} \in \mathbb{R}.$$

For $\mathbb{R} = \mathbb{R} \times \mathbb{R}$, the continuous wavelet transform of $\mathbb{R} \in (\mathbb{R})$ is defined as

$$\begin{aligned} (.,.) &= (.,.) \\ (3.3) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) \overline{g(y)} dx dy = (f, g), \end{aligned}$$

where

$$(f, g) = \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx, \quad (f, f) = \|f\|^2, \quad \|f\| > 0.$$

Here we used the function (f, g) instead of (f, g) because it simplifies the problem of estimating the asymptotic decay properties of the continuous wavelet transform. A fundamental property of this transform is its ability to characterize the local regularity of functions.

4. Wavelet Shrinkage

Wavelet shrinkage became popular the work of Donoho and Johnstone [10]. The general idea is to transform the data to a representation that allows reducing noise in a straight forward way, namely by diminishing the modulus of the wavelet coefficients. Especially the low computational complexity of the wavelet transform has made such approaches highly interesting for signal and image processing applications. A typically nonlinear – shrinkage function

: $\mathbb{R} \rightarrow \mathbb{R}$ is applied to the wavelet transform $W(f)$. Some assumptions on η are $\eta > 0 \Rightarrow \eta(x) \geq 0$, $\eta(-x) = \eta(x)$ and $|\eta(x)| \leq |x|$ for all $x \in \mathbb{R}$. usually η depends on a parameter λ which determines the amount of shrinkage. This parameter is omitted here to simplify the notations.

It is significant to mentioned here that the wavelet transform W is an isometric map from $(L^2(\mathbb{R}), \langle \cdot, \cdot \rangle)$ to $(L^2(\mathbb{R}), \langle \cdot, \cdot \rangle)$. Thus on the subspace given by the image of W , the adjoint operator W^* is the inverse of W .

Thus, a function can be reconstructed from its wavelet transform by means of the formula

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(f)(a, b) \overline{\psi(a(x-b))} da db \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(f)(a, b) \overline{\psi(a(x-b))} da db \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(f)(a, b) \overline{\psi(a(x-b))} da db. \end{aligned}$$

The continuous wavelet transform (3.3) is able to describe the local regularity of functions and distribution and detect the location of singularity points through its decay at fine scale it does not provide additional information about the geometry of the set of singularities. In order to achieve this addition capability, we have to consider wavelet transforms associated with more general dilation group.

Let G be the subgroup of $(\mathbb{R}, +)$ of the matrices

$$g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \quad a > 0, b \in \mathbb{R}.$$

and let us consider the corresponding generalized continuous wavelet transform

$$(\cdot, \cdot) = \int_{\mathbb{R}^2} f(x, y) \overline{g(x, y)} dx dy$$

$$= \int_{\mathbb{R}^2} f(x, y) \overline{g(x, y)} dx dy$$

$$= \int_{\mathbb{R}^2} f(x, y) \overline{g(x, y)} dx dy, \quad \text{where } f, g \in L^2(\mathbb{R}^2)$$

$$\text{where } f, g \in L^2(\mathbb{R}^2) \text{ and } \int_{\mathbb{R}^2} f(x, y) dx dy = 0$$

1 We can verify the factorization $\phi(x, y) = \phi_1(x) \phi_2(y)$, the product of an isotropic and an anisotropic

$$\phi_1(x) = \phi_2(y)$$

dilation matrix and a shear matrix. Thus, the analyzing wavelet $\psi(x, y)$ associated with this function is

transform ranges over various scalers, orientations, and locations controlled by the variables a, b, θ , respectively. The transform $(\cdot, \cdot)$ is called the continuous shearlet transform of f . This transform is able to detect not only the location of singularity points through its decay at fine scales but also the geometric information of the singularity set.

In a similar manner a function can be reconstructed from its shearlet transform by means of the formula

$$f(x, y) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^+} \int_{\mathbb{R}} \hat{f}(a, b, \theta) \psi(x, y; a, b, \theta) da db d\theta$$

$$= \int_{\mathbb{R}^2} \int_{\mathbb{R}^+} \int_{\mathbb{R}} \hat{f}(a, b, \theta) \psi(x, y; a, b, \theta) da db d\theta \quad (4.1)$$

$$= \int_{\mathbb{R}^2} \int_{\mathbb{R}^+} \int_{\mathbb{R}} \hat{f}(a, b, \theta) \psi(x, y; a, b, \theta) da db d\theta$$

5. Continuous Shearlet Transform and the Sum of Smoothed Partial

Derivative Operators

Definition 5.1 [18]. The mother wavelet ψ has fast decay if for any exponent ϵ there exists a constant C_ϵ such that for all $x \in \mathbb{R}^2$,

$$|\psi(x)| \leq \frac{C_\epsilon}{1 + |x|^\epsilon}.$$

Definition 5.2 [18]. The mother wavelet ψ has (a finite number $N \in \mathbb{N}$) order vanishing moments if $\int_{\mathbb{R}^2} \psi(x) x^\alpha dx = 0$ for $|\alpha| \leq N$, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2$, $0 \leq |\alpha| < \infty$.

Lemma 5.1 [16]. A function f is bounded and times continuously partial differentiable with bounded partial derivatives if

$$\int (1 + |x|)^{-\alpha} dx < \infty.$$

If the mother wavelet $\psi(x)$ satisfies Definitions 5.1 and 5.2 we get the following theorem about the shearlet transform and the smoothed partial operators.

Theorem 5.1. Wavelet $\psi(x)$ with a fast decay has a vanishing moment if and only if there exist a group of functions $\{\psi_j(x) | j = 1, 2, \dots\}$ with fast decay such that

$$\psi_j(x) = (-1)^j \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k \psi(x), \quad (5.1)$$

As a consequence,

$$(\psi, \psi_j) = \frac{1}{j!} \int x^j \psi(x) dx, \quad (5.2)$$

with

$$\psi_j(x) = \frac{1}{j!} \int x^j \psi(x) dx.$$

It has no more than vanishing moments if and only if

$$(\psi, \psi_j) \neq 0, \quad |j| = 0, 1, \dots, \infty.$$

Proof. The proof of the theorem follows on the lines of proof of [(16), Thm. A].

6. Two Dimensional Wavelet Shrinkage and Evolution Equation

In this section we shall study the relationship between 2-dimensional wavelet shrinkage and an integrodifferential evolution equation involving the data on a continuous spectrum of scales. Using (5.1), we have

$$\psi_j(x) = \frac{1}{j!} \int x^j \psi(x) dx, \quad \psi_j(x) = \frac{1}{j!} \int x^j \psi(x) dx.$$

and

$$\psi_j(x) = \frac{1}{j!} \int x^j \psi(x) dx, \quad \psi_j(x) = \frac{1}{j!} \int x^j \psi(x) dx.$$

So, for $n = 2$, shearlet translation (ψ_j, ψ_k) and the adjoint transform $\psi_j^*(\psi_k)$ are expressed as

$$(\psi_j, \psi_k) = \int \psi_j(x) \psi_k(x) dx, \quad (6.1)$$

$$\psi_j^*(\psi_k) = \int \psi_j(x) \psi_k(x) dx = \int \psi_j(x) \psi_k(x) dx. \quad (6.2)$$

In the last few years, we have seen an explosion of activity in machine learning, data analysis, and search, implying that similar ideas and concepts, inspired by signal processing weight carry as much power in the context of the orchestration of massive high dimensional data sets.

This digital data, medical records, music, sensor data, financial data etc., can be structured into geometries that result in new organizations of language. As an application, this work is useful in the oil exploration and mining industry, in which one needs to decide where to drill or mine to greatest advantage for finding oil, gas, copper or other minerals. In medical diagnostics, important information can be learned from the analysis of data obtained from radiological, histological, chemical tests and this is important for arriving at an early detection of potentially dangerous tumors and other pathologies (see [5]).

References

K. Bredies, D.A. Lorenz and P. Maass, *Mathematical concepts of multiscale smoothing*, Appl. Comput. Harmonic Anal.

19(2005), 141-161.

A.P. Caldern *Intermediate spaces and interpolation, the complex method*, Studia Math. 24 (1964), 113-190.

E.J. Cands and D.L. Donoho, *Ridgelets: The key to high dimensional intermittency*, Philos Trans. Roy. Soc. Lond. Ser. A 357(1999), 2495-2509.

E.J. Cand and D.L. Donoho, *New tight frames of curvelets and optimal representations of objects with singularities*, Comm. Pure Appl. Math. 57 (2004), 219-266.

R.R. Coifman and M. Gavish, *Harmonic analysis of digital data bases, in Wavelets and Multiscale Analysis: Theory and Applications*, Birkhuser, 2011, J. Cohen and A.I. Zayed (eds.), pp. 161-197.

D.L. Donoho, *Wdgelets: Nearly – minimax estimation of edges*, Ann. Stat. 27(1999), 859- 897.

D.L. Donoho, *Denoisingvis soft thresholding*, IEEE Trans. Inform. Theory 41(1995), 613-627.

D.L. Donoho, I.M. Johnstone, G. Kerkyacharian and D. Picard, *Density estimation by Wavelet thresholding*, Ann. Stat.

24(1996), 508-539.

D.L. Donoho and I.M. Johnstone, *Minimax estimation via wavelet shrinkage*, Ann. Stat. 26(1998), 879-921.

D.L. Donoho and I.M. Johnstone, *Ideal spatial adaption by wavelet shrinkage*, Biometrika, 81(1994), 425-455.

R.A. Devore and B.J. Lucier, *Wavelets*, ActaNumer. 1(1991), 1-56.

R.A. Devore, *Nonlinear approximation*, ActaNumer. 7(1998), 51-150.

- S. Didas and J. Weickert, *Integro – differential equations for continuous multiscale wavelet shrinkage*, Inverse Prob. Imag. 1(2007), 47-62.
- K. Guo and D. Labate, *Optimally sparse multidimensional representation using shearlets*, SIAM J. Math. Anal. 9 (2007), 298-318.
- K. Guo, W – Q. Lim, D. Labate, G. Weiss and E. Wilson, *Wavelets with composite dilations and their MRA properties*, Appl. Comput. Harmon. Anal. 20(2006), 231-249.
- G. Lui, X. Feng and M. Li, *Relationship of d-dimensional continuous multiscale wavelet shrinkage with integro – differential equations*, Chaos Solitons and Fractals, 40 (2009), 1118-1126.
- D.A. Lorenz, *Wavelet shrinkage in signal image processing*, Ph.D.Dissertation Bremen University, 2004, p. 1-99.
- Y. Meyer, *Wavelets and Operators [You Z. Trans]*, Beijing: World Publishing Company, 1992, p. 97-114.
- P. Mrazek, J. Weickert and G. Steidl, *Diffusion inspired shrinkage functions and stability results for wavelets denoising*, Int. J. Comput. Vision, 64(2005), 171-186